

On Discrete Symmetries in $su(2)$ and $su(3)$ Affine Theories and Related Graphs

S. Liénart^{*}, P. Ruelle[†] and O. Verhoeven[‡]

Institut de Physique Théorique
Université Catholique de Louvain
B-1348 Louvain-La-Neuve, Belgium

Abstract

We classify the possible finite symmetries of conformal field theories with an affine Lie algebra $\widehat{su}(2)$ and $\widehat{su}(3)$, and discuss the results from the perspective of the graphs associated with the modular invariants. The highlights of the analysis are first, that the symmetries we found in either case are matched by the graph data in a perfect way in the case of $su(2)$, but in a looser way for $su(3)$, and second, that some of the graphs lead naturally to projective representations, both in $su(2)$ and in $su(3)$.

^{*}Chercheur FRIA

[†]Chercheur Qualifié FNRS

[‡]Collaborateur scientifique FNRS

1 Introduction

Since 1984 [1], two-dimensional conformal field theory has remained a fascinating subject, at the frontier of mathematics and physics. In two-dimensional physics, conformal theories proved to be a powerful tool in the study of critical and off-critical models, and in the dynamics of string theory through the description of the world-sheet. More recently, they have prompted an overwhelming activity in mathematics, in as different subject as the theory of singularities, knot theory, combinatorics, algebraic geometry, von Neumann algebras, ...

Strangely, many of the developments have been triggered by the problem of modular invariance, most notably by the remarkable and not yet fully understood ADE classification of $su(2)$ affine theories. Affine theories are conformal field theories with an affine Lie algebra as symmetry algebra, prime examples being the Wess–Zumino–Witten models (see [2, 3] for recent reviews). Despite much progress and many partial results, only the simpler cases have been fully classified: those based on the $\widehat{su}(2)$ [4, 5] and $\widehat{su}(3)$ [6].

The modular invariants are torus partition functions with periodic boundary conditions in both directions. A better understanding of these models can be gained by allowing other types of monodromies, a systematic source of which being twists effected by the elements of an internal symmetry group. As shown by Zuber [7], this can be done from the knowledge of the critical modular invariants, by examining the modular covariance of the sought twisted partition functions. Illustrations of the method were given in the (A, A) series of Virasoro minimal models, but the analysis was subsequently completed and extended to all minimals models [8], with the somewhat expected results that the symmetry group —discrete— of a model (A, \mathcal{G}) is simply the automorphism group of the Dynkin diagram of \mathcal{G} . This result feeds the general feeling that the graphs that have been associated with modular invariants are key elements in the description of the corresponding conformal theory, a bit like a Dynkin diagram commands a Lie algebra.

It is our purpose to pursue this analysis for affine models, based on the affine algebras $\widehat{su}(2)$ and $\widehat{su}(3)$. For those, we determine the maximal discrete symmetry group and compute the partition functions in all twisted sectors. There are several motivations for doing this.

First of all, no affine model yet has been realized as the scaling limit of a critical lattice model, unlike the unitary minimal Virasoro models. Thus the knowledge of the symmetries can be a useful guide in the search of such models. In addition, the modular invariant partition function contains information on the periodic sector only, and so cannot distinguish between theories with the same partition function but otherwise distinct. Symmetry arguments can do this, or can at least suggest this possibility, as in the $\widehat{su}(3)$ models at level $k = 3$ and 6, see below. Moreover, the presence of a symmetry implies selection rules in the operator product algebra, both in the bulk and on boundaries, resulting in non-trivial constraints on operator product coefficients [9]. And last, the relevance of the graphs may be further probed, by comparing the symmetries found in the affine theory and those of the associated graph(s). From this point of view, we will see that, even though the results are qualitatively different for the $\widehat{su}(2)$ and $\widehat{su}(3)$ models, the graphs are definitely relevant also

for symmetries.

The plan of this article is as follows. In Section 2, we first recall the general setting for discrete symmetries and twisted boundary conditions. We explain the strategy we use in order to determine the maximal symmetry compatible with a given modular invariant, and also the catches of such an analysis. After a few words about orbifolds, we recall the salient features of the association of graphs with modular invariants.

In the following section, we summarize our results, in the form of lists of groups and of partition functions, for all $su(2)$ and $su(3)$ modular invariant theories. We then undertake a general comparison of these results with the graph data. A study of particular cases, which we found the most instructive or representative—all taken from $su(3)$ models—, finishes the third section.

Section 4 is devoted to the proofs. Many of the above results can be proved using the techniques of [8]. So we will content ourselves with giving the explicit proof in two representative series of modular invariants (the type II D -series of automorphism invariants of $su(2)$, and the type I D -series of $su(3)$).

A general summary concluding this work is presented in Section 5.

2 Symmetries and Graphs

2.1 Frustrated partition functions

A particularly interesting geometry to study two-dimensional conformal field theory is the toroidal geometry. It is well-known that two different tori are conformally equivalent if they are related by a modular transformation. Explicitly, if we denote by τ ($\text{Im } \tau > 0$) the standard modulus of the torus, the action of the modular group is given by

$$\left\{ \tau \rightarrow \frac{a\tau + b}{c\tau + d} : a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\} \quad (2.1)$$

This group, isomorphic to $\text{PSL}_2(\mathbb{Z})$, is generated by the two transformations $T : \tau \rightarrow \tau + 1$ and $S : \tau \rightarrow -\frac{1}{\tau}$.

A conformally invariant theory on the torus (i.e., with doubly periodic boundary conditions) is thus described by a modular invariant partition function (MIPF). Here, we consider theories based on untwisted affine Lie algebras [10], which implies that the MIPFs are sesquilinear forms in the characters,

$$Z(\tau) = \sum_{p, p'} [\chi_p(\tau)]^* M_{p, p'} [\chi_{p'}(\tau)], \quad (2.2)$$

for non-negative integral coefficients $M_{p, p'}$. The sum over the character labels p, p' is finite.

The classification of all modular invariant partition functions has been completed for the algebras $\widehat{su}(2)_k$ [4, 5] and $\widehat{su}(3)_k$ [6]. The classification for the other algebras is so far incomplete, which is the reason why we focus in this article on $su(2)$ and $su(3)$.

While MIPFs are associated with periodic boundary conditions, there may be other choices of boundary conditions, corresponding to different monodromy properties of the fields along non-contractible loops. In particular, twisted, non-periodic boundary conditions are possible in a theory which has a symmetry group. The group, G say, acts on the various fields $\phi \rightarrow {}^g\phi$ and leaves the (periodic, bulk) Hamiltonian invariant. The twisted boundary conditions induced by the action of G can be written, in the case of the torus, as

$$\phi(z+1) = {}^g\phi(z), \quad \phi(z+\tau) = {}^{g'}\phi(z), \quad \text{with } g, g' \in G. \quad (2.3)$$

A pair (g, g') defines a specific sector of the theory corresponding to the given twisted boundary conditions. The consistency of the boundary conditions with the translation group of the torus requires that g and g' commute.

Each sector of boundary conditions has its own partition function $Z_{g,g'}$ (called frustrated or twisted in the case of non-periodic b.c.), given, in the Hamiltonian formalism, by [11, 12]

$$Z_{g,g'}(\tau) = \text{Tr}_{\mathcal{H}_g} \left(q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24} g' \right), \quad q = e^{2i\pi\tau}. \quad (2.4)$$

This formula gives a different status to g and g' . \mathcal{H}_g is the Hilbert space of states that live on the fixed time slices (chosen to be lines of constant $\text{Im } z$ in the complex plane), and is subjected to the g boundary condition, while the boundary condition in the time direction is effected by the insertion of g' . Thus g specifies the g -sector \mathcal{H}_g of states/fields of the theory, which carries an action of the centralizer $C(g)$ of g (to which g' belongs). This apparent asymmetry between g and g' cannot exist since there is no preferred direction on the torus. The restoration of the symmetry, based on the modular covariance, is the basis for the determination of the symmetry group of a particular theory.

We have mentioned that g' must be in the centralizer $C(g)$ of g , which thus appears as the symmetry group that remains unbroken in the g -sector (so the periodic sector ($g = e$) has the maximal symmetry, equal to G itself). It is also easy to see that the partition function $Z_{g,g'}$ depends only on the conjugacy class of g' within $C(g)$, and on that of g (within G). Eliminating all redundancies, we obtain a (in general non-square) list of frustrated partition functions $Z_{g,g'}$.

We finish this introductory section by recalling the compatibility conditions that follow from putting together the modular transformations and the existence of a symmetry group.

The presence of the affine symmetry implies that every space \mathcal{H}_g is made up of representations of a pair $\widehat{\mathcal{G}}_k \times \widehat{\mathcal{G}}_k$, with $\widehat{\mathcal{G}}_k$ the level k affine Lie algebra based on \mathcal{G} ($= su(2)$ or $su(3)$ here)

$$\mathcal{H}_g = \bigoplus_{p,p' \in P_{++}^{(n)}} M_{p,p'}^{(g)} (\mathcal{R}_p \otimes \mathcal{R}_{p'}), \quad (2.5)$$

where the numbers $M_{p,p'}^{(g)}$ are multiplicities, *i.e.* non-negative integers. The shifted weights p, p' specify the inequivalent integrable representations of $\widehat{\mathcal{G}}_k$, and the height n is defined by $n = k + h^\vee$, with h^\vee the dual Coxeter number of \mathcal{G} .

Assume now that the theory has a symmetry group G . It means that the generators of the affine Lie algebra (and of the Virasoro algebra) are G -singlets. This in turn implies that the action of $C(g)$ is block-diagonal on inequivalent representations occurring in \mathcal{H}_g , so that, for each pair p, p' , the $M_{p,p'}^{(g)}$ equivalent affine representations carry a representation of $C(g)$. Using the definition of the affine characters $\chi_p(q) = \text{Tr}_{\mathcal{R}_p} q^{(L_0 - c/24)}$, one obtains¹

$$Z_{g,g'}(\tau) = \sum_{p,p' \in P_{++}^{(n)}} M_{p,p'}^{(g)}(g') \chi_p^*(q) \chi_{p'}(q), \quad (2.6)$$

where $M_{p,p'}^{(g)}(g')$ is the character of a unitary representation of $C(g)$ of degree $M_{p,p'}^{(g)} \equiv M_{p,p'}^{(g)}(e)$. As an additional requirement, we will impose that the identity (label $p = \varrho$) appears in the periodic sector \mathcal{H}_e only and is invariant under G , so that $M_{\varrho,\varrho}^{(g)}(g') = \delta_{g,e}$.

For a fixed g' , say of order N , this representation can be fully diagonalized, which then reveals the charges, defined modulo N , of the individual representations

$$M_{p,p'}^{(g)}(g') = \sum_{k=1}^{M_{p,p'}^{(g)}} \zeta_N^{Q(g;p,p';g';k)}, \quad (2.7)$$

with $\zeta_N = e^{2i\pi/N}$. In most cases, the multiplicities $M_{p,p'}^{(g)}$ are equal to 1, so that only one-dimensional representations are involved, from which charges can be read off in a straightforward way.

The frustrated partition functions (2.6) are subjected to strong constraints since they must be compatible with the modular transformations. These leave the torus invariant but mix the two periods, and hence the boundary conditions. The result is the following transformation formula

$$Z_{g,g'}(\tau) = Z_{g,gg'}(\tau + 1) = Z_{g'^{-1},g}(-\frac{1}{\tau}) = Z_{g^a g'^c, g^b g'^d}(\frac{a\tau + b}{c\tau + d}). \quad (2.8)$$

These are severe constraints, which allow to determine the possible symmetries (and all frustrated partition functions) compatible with a given theory (a given MIPF).

In practice, one starts from a supposedly known MIPF $Z_{e,e}$. From it one tries to determine all $Z_{e,g'}$, whose form (in terms of affine characters) is fixed by $Z_{e,e}$, and which themselves yield other functions $Z_{g,g'}$ by modular transformations. This set of functions is finally extended in a maximal way so as to obtain a table of functions $Z_{g,g'}$ which transform in the covariant way prescribed by (2.8). The resulting table corresponds to the maximal symmetry group G that is compatible with the theory specified by the initial MIPF. This procedure has been used in [8] in the case of minimal Virasoro models.

It must be emphasized that we assume throughout that the symmetry group preserves the affine algebra, which implies that all twisted sectors carry representations of the same

¹Provided a proper modification of the stress-energy tensor is made, the same can be done with unrestricted affine characters [13].

symmetry algebra. With this in mind, the non-existence of a symmetry group can mean two things. Either the theory really has no symmetry, or it does have one but which does not preserve the level of the chiral symmetry algebra from which we have chosen to look at the theory, in which case a lower, subalgebra point of view must be taken.

2.2 The determination of the symmetry group

The end of the previous section sketched the way the maximal symmetry group of a theory can be found. Here, we make the procedure a bit more explicit, but more importantly, we discuss the various issues that need to be clarified before the symmetry group can be safely named.

The most obvious question is related to the fact that only pairs (g, g') of commuting elements can be used. In a sense, the non-abelian features of the group cannot all be probed directly.

A good starting point is to classify the cyclic symmetries, and their possible realizations—there can be more than one—, according to the scheme described above. Being abelian groups, there is no restriction on g, g' , and their full structure can be exposed. Since most theories (in this paper) have an abelian cyclic symmetry, this will be the complete story. In a few cases, distinct cyclic symmetries or several realizations of a given cyclic symmetry will be found, indicating the existence of a larger group.

Two cyclic symmetries \mathbb{Z}_N and $\mathbb{Z}_{N'}$ can be assembled into $\mathbb{Z}_{NN'}$ if an action of \mathbb{Z}_N can be consistently defined in the sectors labelled by the elements of $\mathbb{Z}_{N'}$, and vice-versa, i.e. if the two kinds of charges are simultaneously assignable. If not, the two cyclic factors are not commuting subgroups. In this way, one can list the maximal abelian subgroups and look for non-abelian groups that contain them (and only them). In case several groups qualify, one should be able to pick the right one by looking at the various sectors \mathcal{H}_g and the associated data $M_{p,p'}^{(g)}(g')$, which are to be characters of the centralizer $C(g)$.

This procedure presumably leads to a unique group G , as it has to match a number of data: the conjugacy classes and their centralizers, the maximal abelian subgroups, and the character tables of the centralizers (maybe not the whole of them).

Subtleties or difficulties can however be encountered in the above analysis.

1. In looking for a cyclic symmetry, we usually proceed by identifying the way it can be realized in the periodic sector. This yields the functions $Z_{e,g}$ which, by modular transformations, allow to compute the complete table of partition functions $Z_{g,g'}$ for $g, g' \in \mathbb{Z}_N$. When there is a unique function $Z_{e,g}$, this procedure produces a unique table of functions $Z_{g,g'}$. This is however not enough to guarantee that the group \mathbb{Z}_N itself is unique, and indeed it sometimes extends to a power \mathbb{Z}_N^m .

If there is a \mathbb{Z}_N action in the periodic sector by one-dimensional representations (i.e. all fields appear with a degeneracy equal to 1), then $Z_{e,g}$ gives the exact way the

generator of \mathbb{Z}_N acts. So if there is a unique function $Z_{e,g}$, there is also a unique \mathbb{Z}_N action, and a unique \mathbb{Z}_N .

This is no longer the case if higher-dimensional representations are involved, since $Z_{e,g}$ sees only their trace. Suppose that, among others, \mathbb{Z}_N acts in the periodic sector by an M -dimensional representation R_M (a field has multiplicity M ; here, $M \leq 3$ for $su(2)$ and $su(3)$). That the functions $Z_{e,g}$ are unique means that, restricted to that block, any other \mathbb{Z}_N must act by an M -dimensional (distinct) representation R'_M that has the same character as R_M .

So if there is, say, a $\mathbb{Z}_N \times \mathbb{Z}_N$ symmetry, all \mathbb{Z}_N subgroups must act on this M -by- M block with the same character. As it turns out, this implies that M must be large enough for this to happen (as we have seen above, M cannot be 1), with in addition restrictions on the character. For $N = 2$, M must be at least equal to 3, and the character of the representations must be equal to -1 . Written in diagonalized form, the representations of the three \mathbb{Z}_2 subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$ (generated by g_1 and g_2) read : $R(g_1) = \text{diag}(1, -1, -1)$, $R(g_2) = \text{diag}(-1, 1, -1)$ and $R(g_1 g_2) = (-1, -1, 1)$. Likewise, for $N = 3$, the number M must be bigger than 8, and the representations must also have a trace equal to -1^2 .

As mentioned above, M is at most 3 in the models considered here, so that \mathbb{Z}_2 is the only symmetry that can potentially be extended, requiring in addition a trace equal to -1 . But the only model which has a \mathbb{Z}_2 symmetry and a multiplicity equal to 3 is $\widehat{su}(3)_3$ (height 6), for the partition function $D_6 = D_6^*$, where a cyclic symmetry \mathbb{Z}_2 indeed leads to a $\mathbb{Z}_2 \times \mathbb{Z}_2$ group, the two factors being conjugate within a bigger group, namely A_4 . The action is precisely given by the three matrices given above. The details for the determination of the functions $Z_{g_1^k g_2^\ell, g_1^{k'} g_2^{\ell'}}$ are given in Section 3.3.1.

2. The last step in the procedure described above may not work. Indeed the list of all maximal abelian subgroups poses no problem, but there is no guarantee that a single group can accomodate them all. In other words, there may not be a unique maximal symmetry group, but several ones (possibly isomorphic). That this occurs can be taken as an indication that different models exist, which all share the same modular invariant partition function.

The only case where we have seen this situation occur is again in $\widehat{su}(3)_3$, for the partition function $D_6 = D_6^*$. There we found two isomorphic maximal symmetry groups $G = G' = A_4$, the alternating group on four letters. The two groups A_4 have \mathbb{Z}_3 and $\mathbb{Z}_2 \times \mathbb{Z}_2$ as maximal abelian subgroups, but carry a different realization of the \mathbb{Z}_3 factor. The existence of these two symmetry groups is paralleled by the existence of two isospectral graphs, naturally associated to the modular invariant partition function,

²For general N , a solution exists with $M = N^2 - 1$ and again a trace equal to -1 . We suspect that this value of M is minimal.

and reproducing correctly the characters found in the periodic sector \mathcal{H}_e (see Section 3.3.1 for more details).

These elements give some support to the existence of two different models with the same torus partition function $D_6 = D_6^*$.

3. The last issue we want to mention is the possibility that the symmetry group acts on the various sectors by projective representations. This possibility can be easily accounted for if one bears in mind that a projective action modifies the modular transformations of the partition functions.

Let us suppose that G , in fact $C(g)$, acts on the sector \mathcal{H}_g by a projective representation R_g , characterized by a 2-cocycle ω_g :

$$R_g(h)R_g(h') = \omega_g(h, h')R_g(hh'), \quad \forall h, h' \in C(g). \quad (2.9)$$

For such representations, the partition functions $Z_{g,g'}$ transform differently from (2.8), since the composition of group elements is involved. It is not difficult to see that the transformation rules are now

$$Z_{g,g'}(\tau) = \omega_g(g, g') Z_{g,gg'}(\tau + 1) = \omega_{g'}^{-1}(g, g'^{-1}) Z_{g',g'^{-1}}(-\frac{1}{\tau}). \quad (2.10)$$

Let us remark that one should in general expect the various cocycles ω_g to be cohomologically trivial. In the sector \mathcal{H}_g , the action of $C(g)$ is block diagonal with respect to the inequivalent affine representations. So, unless all occurring representations have multiplicity bigger than 1, some of the blocks will correspond to one-dimensional ω_g -projective representations. This implies that ω_g is a coboundary.

Although we were not so much interested in projective representations, they were somehow forced on us in a number of cases, where an expected symmetry, present in the graphs, was not seen to be realized in the field theory, but could however be realized projectively. This situation occurs for a \mathbb{Z}_2 group in all diagonal $su(2)$ models for an odd height (the A_{n-1} , n odd, theories), and the $E_8^{(*)}$ and E_{24} models of $su(3)$. In the $E_8^{(*)}$ models of $su(3)$, it combines with a \mathbb{Z}_3 symmetry to form a projective \mathbb{Z}_6 . To our knowledge, this is the first instance of such a situation.

2.3 Orbifolds

The orbifold construction is well-known (see for instance [14]), and roughly speaking, corresponds to quotienting a theory \mathcal{T} by its symmetry group G (or subgroup). The resulting, orbifold theory \mathcal{T}/G may or may not be different from \mathcal{T} itself.

In some cases, most notably when the group G is abelian, the orbifold theory share the same symmetry as the original one, which can be recovered by orbifolding the orbifold theory. So the two theories, considered with all their twisted sectors, are essentially equivalent.

When G is non-abelian, the symmetry of the orbifold theory is smaller, and equal to the abelianization of G , namely G/G' , where G' is the commutator subgroup [15]. The torus modular invariant partition function of the orbifold theory,

$$Z_{e,e}^{\text{orb}} = \frac{1}{|G|} \sum_{g \in G} \sum_{g' \in C(g)} Z_{g,g'} = \frac{1}{|G|} \sum_{[g]} |[g]| \sum_{g' \in C(g)} Z_{g,g'}, \quad (2.11)$$

collects, from all sectors of the original theory, the fields which are singlets under G ($|[g]|$ is the cardinal of the class of g , and the first summation is over the classes of G).

The formula (2.11) is not the only way to obtain a modular invariant partition function. Distinct orbits under the modular group can be given different coefficients and the result is still modular invariant. That the invariant is a modular invariant partition function puts further restrictions which have been examined in [16]: the different possibilities are classified by the second cohomology group $H^2(G, U(1))$ and lead to what is known as discrete torsion.

Take a 2-cocycle ω on G , and define

$$\epsilon_g(g') = \omega(g, g') \omega(g', g)^{-1}. \quad (2.12)$$

For each g , the function $\epsilon_g(\cdot)$ is a (true) representation of $C(g)$, which can then be used to define the orbifold modular invariant partition function

$$Z_{e,e}^{\text{orb}} = \frac{1}{|G|} \sum_{g \in G} \sum_{g' \in C(g)} \epsilon_g(g') Z_{g,g'}. \quad (2.13)$$

Clearly, this new function corresponds to a projection, in each sector \mathcal{H}_g , onto the fields which transform according to ϵ_g^* .

In this paper, our emphasis is not on the orbifold construction, except that we use it as a cross-check on the frustrated partition functions we find. At the same time, we will compare what the orbifold procedure gives at the level of the graphs, confirming that the graph associated to the orbifold theory is the orbifold graph.

The only model where there is some room for discrete torsion is, once more, the $D_6^{(*)}$ models of $su(3)$, since the symmetry group of all other models has a trivial cohomology H^2 . Details about these models are given in Section 3.3.1, where the discrete torsion yields nothing new compared to the ordinary orbifold construction.

2.4 Graphs

Graphs lie at the heart of the structure of the $su(2)$ theories. Let us recall that each $su(2)$ modular invariant Z can be uniquely associated with a graph Γ , in this case an ADE Dynkin diagram, such that the diagonal terms of Z can be recovered from the graph spectral data [4]. Although the graph does not specify, in a direct way, the whole modular invariant, it does give it implicitly through the modular invariance itself (no distinct modular invariants are known that share the same diagonal terms). It remains a remarkable observation that

the full set of consistent $su(2)$ theories is simply the ADE list. The same observation applies to the Virasoro minimal models, for which the graphs are in fact pairs of graphs, closely related to an A Dynkin diagram and an ADE Dynkin diagram (with a restriction on the number of nodes).

The generalization of these ideas to other models has been considered in [17, 18]. The starting point is always a MIPF, for which one tries to put the diagonal³ terms in correspondence with a graph or a collection of graphs. As shown the most clearly in [13], the connection gets even more significance if one phrases it in terms of an \mathbb{N} -representation of the fusion algebra. We briefly recall the correspondence (a more complete account of the many appearances of graphs in various contexts can be found in [20]).

In a theory with identical chiral and antichiral algebras, let i be a label for the chiral primary fields, with the label 0 for the identity. The primary fields satisfy a fusion algebra with positive integer structure constants $N_{ij}^k = N_{ji}^k$. The matrices $(N_i)_j^k = N_{ij}^k$ themselves satisfy the fusion algebra, and are all diagonalized by the modular S matrix. As a consequence, the eigenvalues of N_i are given by the set $\{S_{ij}/S_{0j}\}_j$.

Let us now assume that Z is the modular invariant partition function of that theory on the torus. If we write the diagonal terms of Z as

$$Z = \sum_{i \in \mathcal{E}} |\chi_i|^2 + \text{non-diagonal}, \quad (2.14)$$

the set \mathcal{E} contains all labels i (with possible multiplicity) such that the periodic sector contains the non-chiral scalar field (i, i) .

Given the set \mathcal{E} , one looks for a \mathbb{N} -representation n_i of the fusion algebra, of dimension $|\mathcal{E}|$, such that the eigenvalues of n_i are determined from \mathcal{E} :

$$\text{spec } n_i = \left\{ \frac{S_{ij}}{S_{0j}} : j \in \mathcal{E} \right\}. \quad (2.15)$$

The regular representation of the fusion algebra, $n_i = N_i$, is always a solution, that corresponds to \mathcal{E} equal to the whole set of available labels, and thus to the diagonal modular invariant.

All matrices n_i have positive integer entries, and so can be viewed as adjacency matrices of graphs (having $|\mathcal{E}|$ nodes). Since the fusion algebra has usually a restricted number of generators, the diagonal structure of Z can eventually be associated with a restricted number of fundamental graphs. For $su(2)$, one generator, hence one graph, is sufficient. For $su(3)$, one needs two generators, and so two (oriented) graphs, but the conjugation implies that the two adjacency matrices are the transpose of each other (the two graphs differ by their orientation). Therefore, both in $su(2)$ and $su(3)$, the information can be condensed in a single graph. We will denote its adjacency matrix by n_f .

Remarkably, all \mathbb{N} -representations are known in the case of the $su(2)$ fusion algebras [17]. At height $n = k+2$, the only (irreducible) solutions correspond to taking for n_f , the generator

³Other couplings than the diagonal ones may be relevant as well [19].

of the fusion ring, the adjacency matrix of all ADE Lie algebras with Coxeter number equal to $n - 1$ or the adjacency matrix of the tadpole graph T with $n - 1$ nodes. Only the ADE diagrams correspond to sets \mathcal{E} which are realized in modular invariants. All representations based on T , and direct sums of ADET diagrams are spurious in the sense that their set \mathcal{E} does not correspond to a modular invariant. Those solutions are thus discarded.

For $su(3)$, the authors of [17] propose a tentative list of graphs (see also [13]). For each modular invariant, they found at least one graph with the right properties, but the novelty, as compared to $su(2)$, is that in a few cases, several isospectral graphs exist. It is presently not known whether this list is exhaustive.

In general, a representation n_i may have a symmetry, in the sense that permutations σ of \mathcal{E} leave the representation invariant

$$(n_i)_{\sigma(j)}^{\sigma(k)} = (n_i)_j^k, \quad \forall i. \quad (2.16)$$

When the representation is given in terms of a single matrix, or one graph, its symmetry group is just the automorphism group of that single graph. In the regular representation, the symmetries of the (fusion) graphs are given by simple currents.

One of our motivations for this work is precisely to see if the relevance of the graphs can be further probed by taking the point of view of symmetries. As far as graphs are concerned, our purpose will be to see if the symmetry data we find in the conformal theories coincide with those we get from the graphs. That question has been answered positively⁴ in the case of the minimal models, both on the torus [8] and on the cylinder [9] where the role played by the graphs in the way boundary conditions behave under the symmetry was investigated. We will see here that it is still largely true in the $su(2)$ and $su(3)$ models, though we found a few cases where the CFT and the graph picture do not quite match. The highlights of this comparison are collected in Section 3.2.

We should close this section by stressing that the connection with graphs goes beyond the diagonal terms of a MIPF, in at least five ways. First the graphs themselves can be used to construct critical lattice models whose continuum limits yield conformal theories which are filiated (via cosets) to the theory that was defining the graph in the first place [21, 22, 17, 18, 23, 24, 25]. (No lattice models however are known that lead in their continuum limit to the affine models themselves.) Second, in the cases of type I (block-diagonal) invariants, a subset of nodes can be found within the graph from which the block structure, hence the full invariant, can be recovered [18]. Third, the graphs leave a trace in some OPE structure constants [26]. Fourth, the non-diagonal pieces of a modular invariant have also received an interpretation in the context of von Neumann factors [27]. And fifth, the graphs and the corresponding fusion representation n_i play a central role on the cylinder, where they classify the possible boundary conditions [28, 29, 30, 13].

⁴But for one class of models: the non-unitary minimal models based on (A_{p-1}, A_{q-1}) for p and q both odd. Strangely, the expected \mathbb{Z}_2 symmetry does not even seem to be realized projectively.

3 Results

This section presents the results of our investigation regarding the symmetries of the $su(2)$ and $su(3)$ affine models. The first subsection makes a list of the symmetry groups for the various theories, as well as their frustrated partition functions, as obtained from the algebraic program described above. In a second part, these results are viewed from the general graph theoretic perspective: without going into the details of all the models, we examine to what extent the symmetry of the theories match that of the graph, and draw general conclusions, mostly based on observations, on what the graphs are capable to say about the conformal theories. The last part is devoted to giving some details about the few cases we found the most instructive. Proofs are relegated to the subsequent section.

We stress the fact that the groups G listed below represent the maximal symmetries which can have a non-projective realization in the corresponding field theories. We have not determined the maximal symmetries with projective realizations. The only projective group actions we have looked at are those connected to automorphisms of the associated graphs Γ . So for a given theory and its graph, we denote by G_{proj} the maximal subgroup of $\text{Aut } \Gamma$ that can be realized projectively.

3.1 Lists

(a) $su(2)$ models

The results we found regarding the symmetries of the $su(2)$ are as follows.

The $\widehat{su}(2)_k$ models, labelled ADE, have a symmetry group exactly equal to the automorphism group of the associated Dynkin diagram, namely no symmetry at all for E_7 and E_8 , the permutation group S_3 for D_4 , and the \mathbb{Z}_2 group in all other cases. When the height $n = k + 2$ is odd however, the \mathbb{Z}_2 symmetry in the diagonal models A_{n-1} can only be realized projectively.

The corresponding frustrated partition functions are given in Table 1. Let us note that the theories $D_{n/2+1}$ for $n = 0 \bmod 4$, and E_6 all have an extended symmetry algebra and a \mathbb{Z}_2 symmetry group. In the E_6 model only does the symmetry group preserve the extended algebra (as easily understood from the conformal embedding $\widehat{su}(2)_{10} \subset \widehat{sp}(4)_1$, the \mathbb{Z}_2 group corresponding to the $\widehat{sp}(4)$ simple currents).

A consistency check can be made on the results of Table 1 by computing orbifold partition functions, given by (2.11). Quotienting by symmetry (sub)groups realized non-projectively, one finds the following relations

$$A_{n-1} \xleftrightarrow{\mathbb{Z}_2} D_{\frac{n}{2}+1} \quad (n \text{ even}), \quad E_6 \xleftrightarrow{\mathbb{Z}_2} E_6, \quad D_4 \xleftrightarrow{\mathbb{Z}_3} D_4, \quad D_4 \xrightarrow{S_3} A_5. \quad (3.1)$$

(b) $su(3)$ models

In analogy with the notation used for $su(2)$, the $su(3)$ invariants are named ADE with the height $n = k + 3$ as a subscript. Because the charge conjugation $S^2 = C$ commutes with

	\mathbb{Z}_2 -frustrated $su(2)$ partition functions ($g^2 = e$)		
A_{n-1} $n \geq 3$	$Z_{e,e} = \sum_{p=1}^{n-1} \chi_p ^2$ $Z_{g,e} = \sum_{p=1}^{n-1} \chi_p^* \chi_{n-p}$	$Z_{e,g} = \sum_{p=1}^{n-1} (-1)^{p+1} \chi_p ^2$ $Z_{g,g} = \sum_{p=1}^{n-1} (-1)^{p+n/2} \chi_p^* \chi_{n-p}$	(Projective realization for n odd)
$D_{\frac{n}{2}+1}$ $n = 4m + 2$	$Z_{e,e} = \sum_{p=1, \text{ odd}}^{2m-1} \chi_p + \chi_{n-p} ^2 + 2 \chi_{2m+1} ^2$ $Z_{g,e} = \sum_{p=2, \text{ even}}^{2m} \chi_p + \chi_{n-p} ^2$	$Z_{e,g} = \sum_{p=1, \text{ odd}}^{2m-1} \chi_p - \chi_{n-p} ^2$ $Z_{g,g} = \sum_{p=2, \text{ even}}^{2m} \chi_p - \chi_{n-p} ^2$	
$D_{\frac{n}{2}+1}$ $n = 4m + 4$	$Z_{e,e} = \sum_{p=1, \text{ odd}}^{n-1} \chi_p ^2 + \sum_{p=2, \text{ even}}^{n-2} \chi_p^* \chi_{n-p}$ $Z_{g,e} = \sum_{p=2, \text{ even}}^{n-2} \chi_p ^2 + \sum_{p=1, \text{ odd}}^{n-1} \chi_p^* \chi_{n-p}$	$Z_{e,g} = \sum_{p=1, \text{ odd}}^{n-1} \chi_p ^2 - \sum_{p=2, \text{ even}}^{n-2} \chi_p^* \chi_{n-p}$ $Z_{g,g} = \sum_{p=2, \text{ even}}^{n-2} \chi_p ^2 - \sum_{p=1, \text{ odd}}^{n-1} \chi_p^* \chi_{n-p}$	
E_6 $n = 12$	$Z_{e,e} = \chi_1 + \chi_7 ^2 + \chi_4 + \chi_8 ^2 + \chi_5 + \chi_{11} ^2$ $Z_{e,g} = \chi_1 + \chi_7 ^2 - \chi_4 + \chi_8 ^2 + \chi_5 + \chi_{11} ^2$ $Z_{g,e} = \chi_4 + \chi_8 ^2 + \left\{ [\chi_1 + \chi_7]^* [\chi_5 + \chi_{11}] + \text{c.c.} \right\}$ $Z_{g,g} = \chi_4 + \chi_8 ^2 - \left\{ [\chi_1 + \chi_7]^* [\chi_5 + \chi_{11}] + \text{c.c.} \right\}$		
	S_3 -frustrated $su(2)$ partition functions ($\omega = e^{2\pi i/3}$)		
D_4 $n = 6$	$Z_{e,e} = \chi_1 + \chi_5 ^2 + 2 \chi_3 ^2$ $Z_{a,e} = \chi_2 + \chi_4 ^2$ $Z_{b,b^k} = \chi_3 ^2 + \omega^k (\chi_1 + \chi_5)^* \chi_3 + \omega^{2k} \chi_3^* (\chi_1 + \chi_5)$	$Z_{e,a} = \chi_1 - \chi_5 ^2$ $Z_{a,a} = \chi_2 - \chi_4 ^2$	$Z_{e,b} = \chi_1 + \chi_5 ^2 - \chi_3 ^2$

Table 1: List of all frustrated partition functions for the $\widehat{su}(2)_k$, $n = k + 2$, affine theories. The affine characters are labelled by shifted weights of $P_{++}^{(n)} = \{1 \leq p \leq n-1\}$. For the D_4 model, the partition functions have been labelled by the S_3 conjugacy classes e , $[a]$ (order 2) and $[b]$ (order 3).

S and T , all modular invariants have a partner $A^*D^*E^*$ obtained by replacing the matrix $M_{p,p'}$ by $M_{p,C(p')} = M_{C(p),p'}$. Four modular invariants are however self-conjugate: $D_6 = D_6^*$, $D_9 = D_9^*$, $E_{12} = E_{12}^*$ and $E_{24} = E_{24}^*$. These equalities have interesting consequences on the symmetries and the graphs.

Conjugate theories have always the same symmetry, since the replacement

$$M_{p,p'}^{(g)}(g') \longrightarrow M_{p,C(p')}^{(g)}(g') \quad (3.2)$$

in all sectors provide a realization of the symmetry in the conjugate theory. One can readily notice that, if this replacement has no effect in the periodic sector of self-conjugate theories, it may not be so in the twisted sectors, where it can give rise to distinct partition functions. As a consequence, self-conjugate modular invariant partition functions may be compatible with two different realizations of a given symmetry group. This indeed happens for the D_6 and D_9 models.

Our results for the symmetries of the $su(3)$ affine models are the following.

All models from the $A_n^{()}$, $D_n^{(*)}$ ($n \neq 6, 9$) series, as well as the exceptionals $E_8^{(*)}$ and $E_{12} = E_{12}^*$ have a \mathbb{Z}_3 symmetry group;*
the self-conjugate model $D_6 = D_6^$ has a symmetry group equal to the alternating group A_4 , which can however be realized in two different ways;*
the other self-conjugate model $D_9 = D_9^$ has \mathbb{Z}_3 as symmetry group, also with two different realizations;*
finally E_{12}^{MS} , E_{12}^{MS} and E_{24} have no symmetry at all.*
Moreover the symmetry of the two models E_8 and E_8^ can be promoted to \mathbb{Z}_6 , and that of E_{24} to the group \mathbb{Z}_2 , provided we allow for projective representations in twisted sectors.*

All frustrated partition functions are given in Tables 2 and 3, while Table 5 collects the projective \mathbb{Z}_6 partition functions in the $E_8^{(*)}$ models. Included in Table 4 is a summary of the symmetry groups.

As explained above, the two realizations of the symmetry in the $D_6^{(*)}$ and $D_9^{(*)}$ models are related to each other by the action of C . Concretely, the difference manifests itself in the sectors twisted by order 3 group elements, meaning, in both cases, that the \mathbb{Z}_3 possesses two different realizations, say \mathbb{Z}_3 and \mathbb{Z}_3' . One easily checks that these two realizations are not compatible with each other, therefore excluding a symmetry $\mathbb{Z}_3 \times \mathbb{Z}_3'$. So, for the $D_9^{(*)}$ models, the symmetry group G should be non-abelian and have two non-conjugate \mathbb{Z}_3 subgroups (and only these), whereas for the $D_6^{(*)}$ models, G must in addition have one $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup. It is not difficult to see, in either case, that such a non-abelian group does not exist (the argument is the same in both cases and is given in Section 3.3.1), and this means that \mathbb{Z}_3 and \mathbb{Z}_3' are parts of two different groups. So the modular invariant partition function $D_9^{(*)}$ is compatible with two realizations of a cyclic \mathbb{Z}_3 group, while $D_6^{(*)}$ is compatible with two realizations of a A_4 group —the only one to contain one \mathbb{Z}_3 and one $\mathbb{Z}_2 \times \mathbb{Z}_2$, up to conjugation, and nothing else.

	\mathbb{Z}_3 -frustrated $su(3)$ partition functions ($g^3 = e$, $\omega = e^{2\pi i/3}$)		
A_n $n \geq 3$	$Z_{e,g^k} = \sum_p \omega^{kt(p)} \chi_p ^2$	$Z_{g,g^k} = \sum_p \omega^{kt(\mu^2(p))} \chi_p^* \chi_{\mu(p)}$	$Z_{g^2,g^k} = \sum_p \omega^{kt(\mu(p))} \chi_p^* \chi_{\mu^2(p)}$
D_n $n \not\equiv 0 \pmod 3$ ($n \geq 5$)	$Z_{e,g^k} = \sum_p \omega^{kt(p)} \chi_p^* \chi_{\mu^{nt(p)}(p)}$ $Z_{g,g^k} = \sum_{t(p)=2n} \chi_p ^2 + \omega^{2kn} \sum_{t(p)=0} \chi_p^* \chi_{\mu(p)} + \omega^{kn} \sum_{t(p)=n} \chi_p^* \chi_{\mu^2(p)}$ $Z_{g^2,g^k} = \sum_{t(p)=n} \chi_p ^2 + \omega^{kn} \sum_{t(p)=0} \chi_p^* \chi_{\mu^2(p)} + \omega^{2kn} \sum_{t(p)=2n} \chi_p^* \chi_{\mu(p)}$		
D_n $n \equiv 0 \pmod 3$ ($n \geq 6$)	$Z_{e,g^k} = \sum_{t(p)=0} \left[\sum_{j=0}^2 \omega^{kj} \chi_p^* \chi_{\mu^j(p)} \right]$ $Z_{g,g^k} = \sum_{t(p)=2} \left[\sum_{j=0}^2 \omega^{kj} \chi_p^* \chi_{\mu^j(p)} \right]$	$Z_{g^2,g^k} = \sum_{t(p)=1} \left[\sum_{j=0}^2 \omega^{kj} \chi_p^* \chi_{\mu^j(p)} \right]$	
E_8 $n = 8$	$Z_{e,g^k} = \chi_{(1,1)} + \chi_{(3,3)} ^2 + \omega^k \chi_{(3,1)} + \chi_{(3,4)} ^2 + \omega^{2k} \chi_{(1,3)} + \chi_{(4,3)} ^2$ $+ \chi_{(4,1)} + \chi_{(1,4)} ^2 + \omega^k \chi_{(2,3)} + \chi_{(6,1)} ^2 + \omega^{2k} \chi_{(3,2)} + \chi_{(1,6)} ^2$ $Z_{g,g^k} = (\chi_{(2,3)}^* + \chi_{(6,1)}^*)(\chi_{(1,6)} + \chi_{(3,2)}) + (\chi_{(3,1)}^* + \chi_{(3,4)}^*)(\chi_{(1,3)} + \chi_{(4,3)})$ $+ \omega^k \left[(\chi_{(1,6)}^* + \chi_{(3,2)}^*)(\chi_{(1,1)} + \chi_{(3,3)}) + (\chi_{(1,3)}^* + \chi_{(4,3)}^*)(\chi_{(1,4)} + \chi_{(4,1)}) \right]$ $+ \omega^{2k} \left[(\chi_{(1,1)}^* + \chi_{(3,3)}^*)(\chi_{(2,3)} + \chi_{(6,1)}) + (\chi_{(1,4)}^* + \chi_{(4,1)}^*)(\chi_{(3,1)} + \chi_{(3,4)}) \right]$ $Z_{g^2,g^k} = (\chi_{(1,3)}^* + \chi_{(4,3)}^*)(\chi_{(3,1)} + \chi_{(3,4)}) + (\chi_{(1,6)}^* + \chi_{(3,2)}^*)(\chi_{(2,3)} + \chi_{(6,1)})$ $+ \omega^{2k} \left[(\chi_{(2,3)}^* + \chi_{(6,1)}^*)(\chi_{(1,1)} + \chi_{(3,3)}) + (\chi_{(3,1)}^* + \chi_{(3,4)}^*)(\chi_{(1,4)} + \chi_{(4,1)}) \right]$ $+ \omega^k \left[(\chi_{(1,1)}^* + \chi_{(3,3)}^*)(\chi_{(1,6)} + \chi_{(3,2)}) + (\chi_{(1,4)}^* + \chi_{(4,1)}^*)(\chi_{(1,3)} + \chi_{(4,3)}) \right]$		
E_{12} $n = 12$	$Z_{e,e} = \chi_{(1,1)} + \chi_{(10,1)} + \chi_{(1,10)} + \chi_{(5,2)} + \chi_{(2,5)} + \chi_{(5,5)} ^2 + 2 \chi_{(3,3)} + \chi_{(3,6)} + \chi_{(6,3)} ^2$ $Z_{e,g^k} = \chi_{(1,1)} + \chi_{(10,1)} + \chi_{(1,10)} + \chi_{(5,2)} + \chi_{(2,5)} + \chi_{(5,5)} ^2 - \chi_{(3,3)} + \chi_{(3,6)} + \chi_{(6,3)} ^2$ $Z_{g,g^k} = Z_{g^2,g^{-k}} = \chi_{(3,3)} + \chi_{(3,6)} + \chi_{(6,3)} ^2$ $+ \left[\omega^k (\chi_{(3,3)}^* + \chi_{(3,6)}^* + \chi_{(6,3)}^*)(\chi_{(1,1)} + \chi_{(10,1)} + \chi_{(1,10)} + \chi_{(5,2)} + \chi_{(2,5)} + \chi_{(5,5)}) + \text{c.c.} \right]$		

Table 2: List of frustrated partition functions for affine $\widehat{su}(3)_k$ ($n = k + 3$) theories. The characters are labelled by shifted weights of $P_{++}^{(n)} = \{p = (a, b) : a, b \geq 1, a + b \leq n - 1\}$. The automorphism of $P_{++}^{(n)}$ is $\mu(a, b) = (n - a - b, a)$ and $t(a, b) = a + 2b \pmod 3$ is the triality. The partition functions of the conjugate theories are related to the above ones via the action of the conjugation operator $C(a, b) = (b, a)$.

	A_4 -frustrated $\widehat{su}(3)_3$ partition functions
D_6 $n = 6$	$Z_{e,e} = \chi_{(1,1)} + \chi_{(1,4)} + \chi_{(4,1)} ^2 + 3 \chi_{(2,2)} ^2$ $Z_{e,a} = \chi_{(1,1)} + \chi_{(1,4)} + \chi_{(4,1)} ^2 - \chi_{(2,2)} ^2$ $Z_{e,b^k} = \chi_{(1,1)} + \omega^k \chi_{(1,4)} + \omega^{2k} \chi_{(4,1)} ^2 \quad (k = 1, 2)$ $Z_{a,e} = 2 \chi_{(2,2)} ^2 + \left[\chi_{(2,2)}^* (\chi_{(1,1)} + \chi_{(1,4)} + \chi_{(4,1)}) + \text{c.c.} \right]$ $Z_{a,a} = 2 \chi_{(2,2)} ^2 - \left[\chi_{(2,2)}^* (\chi_{(1,1)} + \chi_{(1,4)} + \chi_{(4,1)}) + \text{c.c.} \right]$ $Z_{a,a'} = \chi_{(2,2)}^* (\chi_{(1,1)} + \chi_{(1,4)} + \chi_{(4,1)}) - \text{c.c.}$ $Z_{a,aa'} = -\chi_{(2,2)}^* (\chi_{(1,1)} + \chi_{(1,4)} + \chi_{(4,1)}) - \text{c.c.} = -Z_{a,a'}$ $Z_{b,e} = \chi_{(1,2)} + \chi_{(2,3)} + \chi_{(3,1)} ^2$ $Z_{b,b^k} = \chi_{(1,2)} + \omega^k \chi_{(2,3)} + \omega^{2k} \chi_{(3,1)} ^2 \quad (k = 1, 2)$ $Z_{b^2,e} = \chi_{(2,1)} + \chi_{(3,2)} + \chi_{(1,3)} ^2$ $Z_{b^2,b^k} = \chi_{(2,1)} + \omega^{2k} \chi_{(2,3)} + \omega^k \chi_{(1,3)} ^2 \quad (k = 1, 2)$ <p>The second realization D_6^* is obtained from above by the insertion of C</p>

Table 3: List of A_4 partition functions for the $D_6 = D_6^*$ model of $\widehat{su}(3)_3$ (height $n = 6$). Characters are labelled by shifted weights, and partition functions by the four classes of A_4 : in the standard notation of permutations in terms of cycles, $[a] = (\cdot\cdot)(\cdot\cdot)$, and the two classes $[b], [b^2]$ correspond to $(\cdot\cdot\cdot)(\cdot)$. The centralizer of the class $[a]$ is $C(a) = \langle a, a' \rangle \sim \mathbb{Z}_2 \times \mathbb{Z}_2$ which contains all elements of order 2 and the identity. The centralizers of the other two classes are $C(b) = C(b^2) = \langle b \rangle$, isomorphic to \mathbb{Z}_3 .

This strongly suggests that there are two pairs of models (D_6, D_6^*) and (D_9, D_9^*) rather than two models. The models within each pair share the same periodic partition function on the torus and the same symmetry group, but differ in the field content of their \mathbb{Z}_3 -twisted sectors. The same suggestions, based on the existence of isospectral graphs, have been made in [13].

With the help of Tables 2 and 3, one can establish that the orbifold procedure yields the following mappings between the various models (we mean orbifold with respect to the symmetry (sub)groups which are realized non-projectively):

$$A_n \xleftrightarrow{\mathbb{Z}_3} D_n \quad A_n^* \xleftrightarrow{\mathbb{Z}_3} D_n^* \quad (\text{all } n) \quad (3.3)$$

$$D_6^{(*)} \xleftrightarrow{\mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2} D_6^{(*)} \quad D_6 \xrightarrow{A_4} A_6 \quad D_6^* \xrightarrow{A_4} A_6^* \quad (3.4)$$

$$E_8 \xleftrightarrow{\mathbb{Z}_3} E_8^* \quad E_{12} \xleftrightarrow{\mathbb{Z}_3} E_{12}^* \quad (3.5)$$

3.2 Comparison with graphs

There is a number of issues that one can address if one views the above results in a graph theoretic light. We will specifically focus on three issues: the symmetry group itself, the action of G in the various sectors and the orbifold procedure.

The most basic question concerns the identification of the symmetry group of an affine model with the automorphism group of its graph(s). The answer is mostly positive for the $su(2)$ models: if one insists on non-projective realizations, the symmetry of the field theory is exactly the automorphism group of its (Dynkin) graph except in the infinite series of odd level diagonal theories A_{n-1} (n odd) which have no symmetry (though the graph has a \mathbb{Z}_2 automorphism). If we allow projective realizations, the identification of the two groups holds for all theories. This difference in the odd level diagonal theories leave a trace in their minimal models, as it was already observed in [9] that the conformal models based on a pair of A-algebras, both of even rank, do not have a non-projective realization of the (\mathbb{Z}_2) symmetry of their graphs.

The situation for the $su(3)$ theories, summarized in Table 4, is not as neat. One can see that, in the majority of cases, the symmetry of the graph does not coincide with that of the corresponding theory. That statement must be qualified for the two series \mathcal{A}_n^* and $\mathcal{D}_{n \neq 0(3)}$, and also for the isolated model \mathcal{E}_8^* , since they are quotients of graphs by a symmetry group that acts freely (fold graphs). In such cases one naturally expects that this explicit symmetry disappear (while some other can emerge). In all other cases (the 3-colourable graphs), the symmetry group G of the field theory is a subgroup of the automorphism group of its graph(s), of index 1, 2, 4 or even 6 in one case. The situation does not improve much by allowing projective realizations since only in the $E_8^{(*)}$ and E_{24} models does the symmetry group get enlarged (from \mathbb{Z}_3 to \mathbb{Z}_6 , or from $\{e\}$ to \mathbb{Z}_2).

On the question of how G acts on the fields, the answer is universal and strengthens the connection with the graphs recalled in Section 2.4 (see [9] for similar statements in the minimal models). Whenever $G \subset \text{Aut } \Gamma$ (i.e. all graphs for $su(2)$ and all 3-colourable ones for $su(3)$), it turns out that, in addition to encoding the diagonal terms of the periodic partition function, the graph also specifies the characters of the representations by which G acts in the periodic sector. Indeed G has two different actions. On the field theoretic side, it acts on the affine representations labelled (p, p) that occur in the periodic Hilbert space. On the graph theoretic side, being a subgroup of $\text{Aut } \Gamma$, it acts on the eigenspaces V_i of the adjacency matrix n_f of the graph, by some representation R_i . Then the fact is that the two actions coincide (their characters are equal): for all $g \in G \subset \text{Aut } \Gamma$, one has

$$\left\{ \begin{array}{l} {}^g V_i = R_i(g) V_i \\ \text{Tr } R_i(g) = \lambda_i(g) \end{array} \right\} \iff Z_{e,g} = \sum_{i \in \mathcal{E}} \lambda_i(g) |\chi_i|^2 + \text{non-diagonal}. \quad (3.6)$$

This extends to $g \neq e$ the correspondence recalled in Section 2.4 between the graphs and the modular invariant partition functions. It should be stressed however that the correspondence (3.6) for $g \neq e$ appears as a bonus, since it was not put in from the start, contrary to the

Γ	\mathcal{A}_n	$\mathcal{A}_{4,2m+1}^*$	$\mathcal{A}_{2m \geq 6}^*$	\mathcal{D}_6	\mathcal{D}_{3m}	$\mathcal{D}_{3m \pm 1}$	\mathcal{D}_6^*	$\mathcal{D}_{n \geq 7}^*$	\mathcal{E}_8	\mathcal{E}_8^*
$\text{Aut } \Gamma$	\mathbb{Z}_3	—	\mathbb{Z}_2	S_4	S_3	—	$A_4 \times \mathbb{Z}_2$	\mathbb{Z}_3	\mathbb{Z}_6	\mathbb{Z}_2
G	\mathbb{Z}_3	\mathbb{Z}_3	\mathbb{Z}_3	A_4	\mathbb{Z}_3	\mathbb{Z}_3	A_4	\mathbb{Z}_3	\mathbb{Z}_3	\mathbb{Z}_3
G_{proj}									\mathbb{Z}_6	\mathbb{Z}_6

cont'd	$\mathcal{E}_{12}^{(1)}$	$\mathcal{E}_{12}^{(2)}$	$\mathcal{E}_{12}^{(3)}$	$\mathcal{E}_{12}^{(4)}$	$\mathcal{E}_{12}^{(5)}$	\mathcal{E}_{24}
	S_3	S_3	$S_3 \times \mathbb{Z}_3$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	—	\mathbb{Z}_2
	\mathbb{Z}_3	\mathbb{Z}_3	\mathbb{Z}_3	—	—	—
						\mathbb{Z}_2

Table 4: Synopsis of the groups of symmetry pertaining to the $su(3)$ affine models and their graphs, in a notation borrowed from [13]. The top line designates the graphs: the superscript i appended to the graph \mathcal{E}_{12} labels three isospectral graph, $\mathcal{E}_{12}^{(4)}$ and $\mathcal{E}_{12}^{(5)}$ correspond respectively to the invariants $E_{12}^{*\text{MS}}$ and E_{12}^{MS} . The second line gives the automorphism group $\text{Aut } \Gamma$ of the various graphs, whereas the third one mentions the group of symmetry G we found in the field theories. The last line refers to the group $\text{Aut } \Gamma \supset G_{\text{proj}} \supset G$ of which the corresponding field theory carries projective representations.

case $g = e$.

To consider and investigate (3.6) for the graph automorphisms g other than those in G is natural, as it yields sensible functions $Z_{e,g}$, from which one can start. For a reason that is not clear to us, they always lead to functions $Z_{g,e}$ with non-integer coefficients, except in four cases, the odd level diagonal $su(2)$ models and the E_8 , E_8^* , E_{24} models of $su(3)$. For these, the graphs have a \mathbb{Z}_2 symmetry which yields through (3.6) sensible functions $Z_{e,g}$ and $Z_{g,e}$, but produces $\pm i$ coefficients (charges) in $Z_{g,g}$. One can easily see that these three functions are fully compatible with a projective action of that \mathbb{Z}_2 in the twisted sector, in the sense of Section 2.2. In the case of the E_8 model, this projective \mathbb{Z}_2 combines with a \mathbb{Z}_3 to form a projective \mathbb{Z}_6 , see Section 3.3.4 for more details.

Thus concretely, (3.6) means that much of the Tables 1,2,3 and 5 can actually be read off from the graphs, since a spectral analysis of their adjacency matrices yield explicit expressions for the numbers $M_{p,p}^{(e)}(g)$. When there are several isospectral graphs for a given modular invariant partition function, the numbers $M_{p,p}^{(e)}(g)$ one gets may or may not differ from graph to graph. For the $su(3)$ invariants $D_6^{(*)}$ or $D_9^{(*)}$, there are two isospectral graphs \mathcal{D}_6 , \mathcal{D}_6^* and \mathcal{D}_9 , \mathcal{D}_9^* which have a different automorphism group, and which yield different values for $M_{p,p}^{(e)}(g)$, which in turn lead to two different realizations of the symmetry. In contrast, there are three graphs $\mathcal{E}_{12}^{(i)}$ associated to the single invariant E_{12} , but all three have a \mathbb{Z}_3 automorphism subgroup which gives the same numbers $M_{p,p}^{(e)}(g)$.

The correspondence (3.6) has a counterpart in the non-periodic, twisted sectors. For the same class of models, such that $G \subset \text{Aut } \Gamma$, the diagonal terms of $Z_{g,e}$ are encoded in the subgraph of Γ made of the nodes that are fixed by g , via the same relation as before. Namely, if $\Gamma^{(g)}$ denotes the subgraph of Γ fixed by g (possibly empty), with adjacency matrix $n_f^{(g)}$, it appears in most cases that

$$\text{spec } n_f^{(g)} = \left\{ \frac{S_{f,j}}{S_{0,j}} : j \in \mathcal{E}_g \right\} \iff Z_{g,e} = \sum_{i \in \mathcal{E}_g} |\chi_i(q)|^2 + \text{non-diagonal}. \quad (3.7)$$

The only cases for which the connection breaks down are the graphs which contain multiple (double) links, namely the \mathcal{D}_{3n} series and the graph $\mathcal{E}_{12}^{(1)}$, all in $su(3)$. All of them have a \mathbb{Z}_3 symmetry, with a fixed point graph which has the correct number of nodes but not the correct eigenvalues. For the graphs \mathcal{D}_{3n} , the sets \mathcal{E}_g and \mathcal{E}_{g^2} are such that $\{S_{f,j}/S_{0,j}\}_{j \in \mathcal{E}_g \text{ or } \mathcal{E}_{g^2}}$ are not even closed under complex conjugation, and so cannot possibly form the spectrum of an integer matrix.

As to the way G (or a proper centralizer) acts in a twisted sector \mathcal{H}_g , in all but one case, one can see from the tables that all diagonal fields in all twisted sectors are invariant under the relevant symmetry group (centralizer). This is again consistent, since the subgraph $\Gamma^{(g)}$ has no symmetries left over from G . The only exception is in the \mathbb{Z}_2 -twisted sector of the D_6 , D_6^* models, see Table 3. The same observation has been made in the minimal conformal models, and was actually a happy fact in the analysis of the corresponding boundary theories, as the nodes of the fixed point graph were put in one-to-one relation with G -invariant boundary conditions [9]. An analogous correspondence is to be expected in the affine models.

The last point of comparison we want to make concerns the orbifold procedure. The results regarding the orbifold partition functions are given in (3.1) for $su(2)$, and in (3.3) to (3.5) for $su(3)$. As cosets of graphs by subgroups of their automorphism groups can be defined, the question naturally arises to see if, starting from a theory \mathcal{T} and graph Γ , the orbifold theory \mathcal{T}/H has a graph that is given by the quotient graph Γ/H (again we restrict to cases where $H \subset G \subset \text{Aut } \Gamma$ is realized non-projectively).

That question is perhaps less significant because it depends on the prescriptions used to define the quotient graph, especially when the subgroup H has fixed points and when the graph has multiple links. However in the simplest cases, when the graph has no multiple links and when the elements of H by which we want to quotient have the same fixed points, standard prescriptions [31] require to multiply all fixed points before proceeding to the quotient. When there are multiple links, natural though ad hoc prescriptions can be given as well regarding the way these links must be split.

With the same restrictions as above regarding the non 3-colourable graphs of $su(3)$, one can check, using these prescriptions, that indeed the quotient graph is the graph of the orbifold theory (obvious for some classes of graphs, constructed by quotient). Interestingly, the conjugate graphs \mathcal{D}_6 and \mathcal{D}_6^* are the orbifold of each other under a quotient by $\mathbb{Z}_2 \subset A_4$, something that cannot be revealed at the level of the partition functions because the two

theories D_6 and D_6^* only differ in their \mathbb{Z}_3 -twisted sectors. Also, the \mathbb{Z}_3 quotient exchanges the $su(3)$ graph $\mathcal{E}_{12}^{(1)}$ and $\mathcal{E}_{12}^{(2)}$, while $\mathcal{E}_{12}^{(3)}$ is a self-orbifold.

The conclusion one should draw from this comparison is that symmetry considerations makes the relationship between the modular invariants and the graphs tighter. The $su(2)$ theories are particularly well-behaved in this respect since the connection for them is complete (exception made of the projective realizations): the graphs give the correct symmetries, the correct twisted partition functions and the correct orbifold relations. To a large extent, the same is true of the $su(3)$ models, since again the graphs are capable to predict much of the symmetries of the corresponding models. But at the same time, the same considerations make this connection fade, as the graphs in many cases would predict more symmetries than what is actually realized. Against this, one could argue that the list of graphs is not exhaustive in $su(3)$ and that yet to be discovered new graphs would restore a perfect connection. Unlikely as it is, one cannot hope this to be strictly true: the two isospectral graphs \mathcal{D}_6 and \mathcal{D}_6^* are the only ones whose spectrum matches the diagonal pieces of the $D_6^{(*)}$ modular invariant, and yet both have a bigger symmetry than A_4 , the maximal symmetry group that can be realized in the field theory(ies).

We finish the presentation of our results by turning to some of the most illustrative and instructive cases, where much of the above results can be seen explicitly.

3.3 Selecta

We give in this section a more detailed study of a few particular cases, worth of being singled out for their peculiarities. Not surprisingly, they all belong to the $su(3)$ graphs.

3.3.1 The $D_6^{(*)}$ invariant of $su(3)$

This is by far the richest and most interesting case, by many aspects, the most obvious one being the size of its symmetry group.

Let us recall that this self-conjugate modular invariant,

$$Z(D_6, D_6^*) = |\chi_{(1,1)} + \chi_{(1,4)} + \chi_{(4,1)}|^2 + 3 |\chi_{(2,2)}|^2, \quad (3.8)$$

has six diagonal terms $|\chi_i|^2$ with ratios $S_{f,i}/S_{0,i} = 2, 2\omega, 2\omega^2, 0, 0, 0$. These numbers are the eigenvalues of the adjacency matrix of two (and only two) graphs, noted \mathcal{D}_6 and \mathcal{D}_6^* (after [13]) and shown in Figure 1.

The automorphism group of \mathcal{D}_6 is S_4 and corresponds to all permutations of the nodes 3 to 6. The three eigenvectors of non-degenerate eigenvalue $2\omega^k$ are invariant under S_4 , while the other three transform in an irreducible representation (with character equal to -1 on order 4 group elements). Restricted to A_4 , it remains irreducible of degree 3, and has character $\lambda_3(g) = (3, -1, 0, 0)$ for respectively $g = e$, $g \in [a]$ (the class of order 2 elements), $g \in [b]$ (first class of order 3 elements) and $g \in [b^2]$ (second class of order 3 elements). Thus,

Figure 1: The isospectral graphs \mathcal{D}_6 (left) and \mathcal{D}_6^* (right). All links are oriented, f.i. from black to grey to white to black.

under the A_4 subgroup of $\text{Aut } \mathcal{D}_6$, the six eigenvectors of the adjacency matrix transform in $r_0 \oplus r_0 \oplus r_0 \oplus r_3$, with r_0 the trivial representation.

The other graph \mathcal{D}_6^* has automorphism group $A_4 \times \mathbb{Z}_2$ (the graph is an (oriented) octaedron with base formed f.i. by the nodes 1,3,4,6). The group is generated by the order 3 rotations and the three commuting transpositions σ_i which exchange the nodes i and $i+3$. The combined $\sigma_1\sigma_2\sigma_3$ (the inversion of the octaedron through the origin) generates the center \mathbb{Z}_2 of the group. The A_4 subgroup is generated by the rotations and the products $\sigma_i\sigma_j$. Under the action of A_4 , the three non-degenerate eigenvectors transform in the three inequivalent one-dimensional representations r_k , $k = 0, 1, 2$, with characters $\lambda_k(g) = (1, 1, \omega^k, \omega^{2k})$ in the same notation as above, while the three degenerate eigenvectors transform again in the degree 3 irreducible representation with character $\lambda_3(g) = (3, -1, 0, 0)$. Thus the six eigenvectors transform under A_4 as $r_0 \oplus r_1 \oplus r_2 \oplus r_3$.

Starting from the partition function (3.8), one first checks that it is compatible with the cyclic symmetries \mathbb{Z}_2 (in a unique way in the periodic sector) and \mathbb{Z}_3 (in two different ways). The \mathbb{Z}_2 partition functions one finds are those called $Z_{e,a}, Z_{a,e}, Z_{a,a}$ in Table 3, from which one sees that the periodic sector has a field three times degenerate, on which the \mathbb{Z}_2 generator acts by a representation of character equal to -1 . As explained in Section 2.2, these are precisely the circumstances under which the \mathbb{Z}_2 group can extend. Writing A, B, C for $Z_{e,a}, Z_{a,e}, Z_{a,a}$, we look for a signed partition function D , such that the following 4-by-4 table makes a consistent set of partitions functions:

$Z_{g,g'}$	e	a	a'	aa'
e	$Z_{e,e}$	A	A	A
a	B	C	D	$T^\dagger DT$
a'	B	$S^\dagger DS$	C	$(ST)^\dagger D(ST)$
aa'	B	$(TS)^\dagger D(TS)$	$(STS)^\dagger D(STS)$	C

Much of it is fixed by the requirement that its restriction to any \mathbb{Z}_2 subgroup yields back

the known 2-by-2 table. It turns out that there is a unique⁵ solution for $D = Z_{a,a'}$, given in Table 3. Thus the group \mathbb{Z}_2 indeed extends to $\mathbb{Z}_2 \times \mathbb{Z}_2$, and to nothing bigger.

So the partition function (3.8) is compatible with one $\mathbb{Z}_2 \times \mathbb{Z}_2$ and two \mathbb{Z}_3 , with corresponding partition functions given in Table 3 (the two \mathbb{Z}_3 realizations are related via the action of C). One easily checks that none of them is compatible with another one, making them non-commuting subgroups of a bigger G . The argument that such a G does not exist is not difficult, and relies on Sylow's theorems.

The two \mathbb{Z}_3 subgroups are 3-groups. Either they are Sylow subgroups, which is impossible since they would be conjugate, contradicting the fact that they realized differently on the periodic Hilbert space. Or the order of G is divisible by 9, in which case there is a 3-subgroup of order 9, which can only be a cyclic \mathbb{Z}_9 or a product $\mathbb{Z}_3 \times \mathbb{Z}_3$. Both possibilities are to be ruled out.

So the two \mathbb{Z}_3 subgroups cannot be accommodated within a single group. Retaining only one of the two leads to the group $G = A_4$, which is therefore the maximal symmetry group. It can be realized in two different ways through the choice of its \mathbb{Z}_3 subgroup, the two realizations being related by conjugation. Moreover, the two realizations one obtains correspond exactly to the A_4 action in the two graphs \mathcal{D}_6 and \mathcal{D}_6^* , explicated above, since they yield respectively (see Table 3)

$$Z_{e,g}(\mathcal{D}_6) = |\chi_{(1,1)}|^2 + |\chi_{(1,4)}|^2 + |\chi_{(4,1)}|^2 + \lambda_3(g) |\chi_{(2,2)}|^2 + \text{non-diagonal}, \quad (3.9)$$

and

$$Z_{e,g}(\mathcal{D}_6^*) = |\chi_{(1,1)}|^2 + \lambda_1(g) |\chi_{(1,4)}|^2 + \lambda_2(g) |\chi_{(4,1)}|^2 + \lambda_3(g) |\chi_{(2,2)}|^2 + \text{non-diagonal}, \quad (3.10)$$

in terms of A_4 irreducible characters. These facts provide the basis for our earlier suggestion that the partition function (3.8) corresponds to two different models.

We also note that the (fixed points of the) graph gives the correct number of diagonal terms in the frustrated partition functions: 3 or 0 in \mathbb{Z}_3 -twisted sector and 2 in the \mathbb{Z}_2 -twisted one.

Finally, the orbifolds, both at the level of the field theories and at the level of the graphs, have been discussed earlier. The last point we want to comment on concerns the possibility of making a twisted orbifold, through discrete torsion. The procedure has been recalled in Section 2.3.

The introduction of discrete torsion requires a (cohomologically) non-trivial 2-cocycle ω on the orbifold group. As $H^2(A_4, U(1)) = \mathbb{Z}_2$, there is a unique choice for ω . To this ω corresponds the central extension (double covering) $\text{SL}_2(\mathbb{F}_3)$ of $A_4 \sim \text{PSL}_2(\mathbb{F}_3)$. The character table of $\text{SL}_2(\mathbb{F}_3)$ (tabulated f.i. in [32]) provides the projective and non-projective characters of A_4 . One can see that A_4 has three projective representations, all of degree 2, with zero characters on the class $[a]$. So only for the three elements a, a', aa' of this

⁵Up to an overall sign, but since $T^\dagger DT = -D$, it reflects the ambiguity in the choice of the generator of the second \mathbb{Z}_2 factor (a' or aa').

class can the quantity $\epsilon(g, g')$ be different from 1⁶. It can be most conveniently computed from one explicit representation R , since for commuting elements, one has $R(g)R(g') = \epsilon(g, g')R(g')R(g)$. Choosing any two-dimensional representation of $\text{SL}_2(\mathbb{F}_3)$ given in [32], one finds

$$\epsilon(g, g') = \begin{cases} +1 & \text{if } g, g' \notin \{a, a', aa'\} \text{ or if } g = g', \\ -1 & \text{if } g \neq g' \in \{a, a', aa'\}. \end{cases} \quad (3.11)$$

Thus in the sum (2.13) giving the orbifold partition function, the part that concerns the sector \mathcal{H}_a reads $Z_{a,e} + Z_{a,a} - Z_{a,a'} - Z_{a,aa'}$ whereas the usual orbifold summation would take an all plus combination. This makes however no difference since $Z_{a,a'} = -Z_{a,aa'}$. So with discrete torsion or not, the orbifold of the D_6 (resp. D_6^*) model by its A_4 symmetry is the model A_6 (resp. A_6^*) with symmetry \mathbb{Z}_3 , equal, as expected, to the quotient of A_4 by its commutator subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$.

3.3.2 The $D_9^{(*)}$ invariant of $\text{su}(3)$

This case is similar to the previous case except that the symmetry is smaller. All that has been said for the two \mathbb{Z}_3 realizations in the $D_6^{(*)}$ can be repeated here. In particular there is no group G that can accomodate them both, so that the same argument points to the existence of two separate field theories, D_9 and D_9^* , with the same torus partition function.

3.3.3 The E_{12} invariant of $\text{su}(3)$

This is the last of the three cases where several isospectral graphs, here three, are known to correspond to the same modular invariant. The three graphs, noted $\mathcal{E}_{12}^{(i)}$ in [13], are shown in Figure 2.

Figure 2: The three isospectral graphs $\mathcal{E}_{12}^{(1)}$ (left), $\mathcal{E}_{12}^{(2)}$ (middle) and $\mathcal{E}_{12}^{(3)}$ (right), corresponding to the self-conjugate modular invariant E_{12} .

⁶The theory of projective representations [33] says that the elements g of all other classes are ω -regular, meaning precisely that $\omega(g, g') = \omega(g', g)$ for all $g' \in C(g)$.

The field theory has a maximal \mathbb{Z}_3 symmetry, with partition functions (see Table 2)

$$Z_{e,g^k} = |\chi_{(1,1)} + \chi_{(10,1)} + \chi_{(1,10)} + \chi_{(5,2)} + \chi_{(2,5)} + \chi_{(5,5)}|^2 + (\omega^k + \omega^{2k}) |\chi_{(3,3)} + \chi_{(3,6)} + \chi_{(6,3)}|^2. \quad (3.12)$$

with ω a primitive third root of 1. It is a self-orbifold theory. One would like to see if these facts, compared with the graph data, can help select one of the three graphs.

Their automorphism group is S_3 , S_3 and $S_3 \times \mathbb{Z}_3$ respectively. In $\mathcal{E}_{12}^{(1)}$, the S_3 simply permutes the three wings attached to the central axis. In $\mathcal{E}_{12}^{(2)}$, the order 3 automorphisms are rotations around the axis; the order 2 elements are the conjugates of $(4 \leftrightarrow 6, 7 \leftrightarrow 8, 11 \leftrightarrow 12)$. In $\mathcal{E}_{12}^{(3)}$, the factor \mathbb{Z}_3 are the rigid rotations of the graph, and the S_3 permutes the three nodes of each peripheral group, the same way within each group.

In each case, the unique, up to conjugation, \mathbb{Z}_3 subgroup of S_3 acts on the eigenvectors of their adjacency matrix in the way shown by the diagonal terms of (3.12). The third graph $\mathcal{E}_{12}^{(3)}$ is its own orbifold under the \mathbb{Z}_3 subgroup of S_3 , whereas the first two are the orbifold of each other, provided the double links of $\mathcal{E}_{12}^{(1)}$ are handled properly. The fixed point graphs are all equal to an oriented triangle (again with an ad hoc prescription for the double links), whose adjacency matrix has the three third roots of 1 as eigenvalues. These are also the values of the ratios $S_{f,i}/S_{0,i}$ for the three diagonal terms in the twisted sectors, $i = (3,3), (3,6), (6,3)$. Thus the only feature that distinguishes them is the fact the third one is its own orbifold. Esthetically, this property may seem desirable as the field theory is its own orbifold too, but the argument is not compelling.

Other methods [26, 34, 27] point to $\mathcal{E}_{12}^{(1)}$ as the graph that is genuinely associated with the E_{12} theory.

3.3.4 The E_8 invariant of $\mathfrak{su}(3)$

The graph corresponding to the $\widehat{\mathfrak{su}}(3)_5$ invariant E_8 is shown in Figure 3. Its automorphism group \mathbb{Z}_6 acts by rigid rotations.

Figure 3: The graph \mathcal{E}_8 (left) corresponding to the level 5 $\mathfrak{su}(3)$ modular invariant E_8 , and the conjugate graph \mathcal{E}_8^* (right).

Its \mathbb{Z}_3 subgroup is the maximal symmetry that has a non-projective realization in the field theory, and leads to the partition functions displayed in Table 2. As to the \mathbb{Z}_2 subgroup, the graph data and the relation (3.6) allow to write a unique $Z_{e,g}$ which yields a sensible integer function $Z_{g,e}(\tau) = Z_{e,g}(\frac{-1}{\tau})$. The T transform of the latter, usually equal to $Z_{g,g}$, acts

$n = 8$	Projective \mathbb{Z}_6 -frustrated $\widehat{su}(3)_5$ partition functions
E_8	$Z_{e,g^k} = \chi_0 ^2 + \zeta^{2k} \chi_1 ^2 + \zeta^{4k} \chi_2 ^2 + \zeta^{6k} \chi_3 ^2 + \zeta^{8k} \chi_4 ^2 + \zeta^{10k} \chi_5 ^2$ $Z_{g,g^k} = \zeta^{7k} \chi_0^* \chi_1 + \zeta^{9k} \chi_1^* \chi_2 + \zeta^{11k} \chi_2^* \chi_3 + \zeta^k \chi_3^* \chi_4 + \zeta^{3k} \chi_4^* \chi_5 + \zeta^{5k} \chi_5^* \chi_0$ $Z_{g^2,g^k} = \zeta^{2k} \chi_0^* \chi_2 + \zeta^{4k} \chi_1^* \chi_3 + \zeta^{6k} \chi_2^* \chi_4 + \zeta^{8k} \chi_3^* \chi_5 + \zeta^{10k} \chi_4^* \chi_0 + \chi_5^* \chi_1$ $Z_{g^3,g^k} = \zeta^{9k} \chi_0^* \chi_3 + \zeta^{11k} \chi_1^* \chi_4 + \zeta^k \chi_2^* \chi_5 + \zeta^{3k} \chi_3^* \chi_0 + \zeta^{5k} \chi_4^* \chi_1 + \zeta^{7k} \chi_5^* \chi_2$ $Z_{g^4,g^k} = \zeta^{4k} \chi_0^* \chi_4 + \zeta^{6k} \chi_1^* \chi_5 + \zeta^{8k} \chi_2^* \chi_0 + \zeta^{10k} \chi_3^* \chi_1 + \chi_4^* \chi_2 + \zeta^{2k} \chi_5^* \chi_3$ $Z_{g^5,g^k} = \zeta^{11k} \chi_0^* \chi_5 + \zeta^k \chi_1^* \chi_0 + \zeta^{3k} \chi_2^* \chi_1 + \zeta^{5k} \chi_3^* \chi_2 + \zeta^{7k} \chi_4^* \chi_3 + \zeta^{9k} \chi_5^* \chi_4$ with $\chi_0 = \chi_{(1,1)} + \chi_{(3,3)} \quad \chi_1 = \chi_{(1,3)} + \chi_{(4,3)}$ $\chi_2 = \chi_{(2,3)} + \chi_{(6,1)} \quad \chi_3 = \chi_{(1,4)} + \chi_{(4,1)}$ $\chi_4 = \chi_{(1,6)} + \chi_{(3,2)} \quad \chi_5 = \chi_{(3,1)} + \chi_{(3,4)}$
E_8^*	All partition functions are obtained from the above via C

Table 5: Consistent set of partition functions for the E_8 and E_8^* models, frustrated by a \mathbb{Z}_6 group of symmetry, realized projectively. The number ζ is a primitive 12-th root of unity.

on the various fields with phases $\pm i$, which indeed suggest a projective action. Combined with the \mathbb{Z}_3 , one expects a projective action of \mathbb{Z}_6 . Let us show that it is indeed the case.

Let g be a generator of \mathbb{Z}_6 . The graph data provide a specific form for the diagonal terms of $Z_{e,g}$, which one tries to complete so as to get a integer function $Z_{g,e}(\tau) = Z_{e,g}(\frac{-1}{\tau})$. As before, this fixes it uniquely to (with ζ a primitive twelveth root of 1)

$$\begin{aligned}
Z_{e,g} = & |\chi_{(1,1)} + \chi_{(3,3)}|^2 + \zeta^2 |\chi_{(1,3)} + \chi_{(4,3)}|^2 + \zeta^4 |\chi_{(2,3)} + \chi_{(6,1)}|^2 + \zeta^6 |\chi_{(1,4)} + \chi_{(4,1)}|^2 \\
& + \zeta^8 |\chi_{(1,6)} + \chi_{(3,2)}|^2 + \zeta^{10} |\chi_{(3,1)} + \chi_{(3,4)}|^2.
\end{aligned} \tag{3.13}$$

If one performs modular transformations on it, one finds that the action of g is by sixth roots of unity in the sectors \mathcal{H}_{e,g^2,g^4} , and by twelveth roots of 1 in \mathcal{H}_{g,g^3,g^5} , more precisely by i times sixth roots of 1, a clear sign that projective representations are present in those three sectors. The projective or non-projective nature of the representations in the various sectors must however obey the consistency conditions set by the modular transformations. One should be able to find six cocycles ω_g , one for each sector which determines the nature of the \mathbb{Z}_6 representations in that sector, such that the transformation laws (2.10) are fulfilled

$$Z_{g,g'}(\tau) = \omega_g(g, g') Z_{g,gg'}(\tau + 1) = \omega_{g'}^{-1}(g, g'^{-1}) Z_{g',g^{-1}}(-\frac{1}{\tau}). \tag{3.14}$$

We note that in general the determination of the cocycles affects that of the partition

functions and vice-versa: the functions $Z_{g,g'}$ contain the full information about the cocycles ω_g , but cannot be computed unless some cocycles are given. As a consequence, more than one consistent set of partition functions and of cocycles can be found (except for a \mathbb{Z}_2 group). One may also observe that a number of partition functions can be determined from Z_{e,g^k} without the knowledge of any cocycle, namely all $Z_{g^k,e}$ and the diagonal ones Z_{g^k,g^k} . Moreover, a limited number of cocycle values are needed to compute the full table of partition functions.

The simplest solution to (3.14) is as follows. As the group acts in the sectors \mathcal{H}_{e,g^2,g^4} by sixth roots of unity, there is no need to introduce a non-trivial cocycle there. So we put $\omega_e = \omega_{g^2} = \omega_{g^4} = 1$. On the other hand, the function $Z_{g,g}$ says that g acts in \mathcal{H}_g by i times sixth roots of unity. The simplest to assume is that g^k acts as the k -th power of g , and this fixes the cocycle ω_g to be $\omega_g(g^k, g^l) = i^{k+l-\langle k+l \rangle_6}$, where $\langle n \rangle_6$ stands for the residue of n modulo 6, taken between 0 and 5. In turn, this allows to compute all Z_{g,g^k} and then all $Z_{g^k,g}$. The same assumption for \mathcal{H}_{g^3,g^5} as for \mathcal{H}_g (the action of g^k is the k -th power of the action of g) yields the same cocycle, so that all together

$$\begin{aligned}\omega_e(g^k, g^l) &= \omega_{g^2}(g^k, g^l) = \omega_{g^4}(g^k, g^l) = +1, \\ \omega_g(g^k, g^l) &= \omega_{g^3}(g^k, g^l) = \omega_{g^5}(g^k, g^l) = i^{k+l-\langle k+l \rangle_6}.\end{aligned}\tag{3.15}$$

They determine non-ambiguously all partition functions, given in Table 5, which display a neat cyclic structure. Finally, one has to make sure that our assumptions are self-consistent by verifying that the transformations (3.14) are satisfied for all g, g' , which they are.

The same analysis can be made for the E_8^* model, starting from the graph \mathcal{E}_8^* . It has a \mathbb{Z}_2 automorphism which leads to a projective \mathbb{Z}_2 symmetry in the corresponding field theory, and eventually to a projective \mathbb{Z}_6 .

3.3.5 The E_{24} invariant of $\text{su}(3)$

This is the third and last invariant of $\text{su}(3)$ which is compatible with a projective symmetry. A numerical analysis, using the Galois symmetry, shows that the field theory is not compatible with a symmetry group acting by true representations. However, guided by the corresponding graph \mathcal{E}_{24} , reproduced below, one can see that it is compatible with a \mathbb{Z}_2 projective action.

The only non-trivial automorphism of the graph is an inversion through its center, and preserves the colour of the nodes. Acting on the eigenvectors of the adjacency matrix, it has twelve eigenvalues equal to $+1$ and twelve eigenvalues equal to -1 . All eigenvalues $+1$ correspond to the twelve characters contained in the block of the identity, from which one sets

$$Z_{e,g} = |\chi_0|^2 - |\chi_1|^2,\tag{3.16}$$

with

$$\begin{aligned}\chi_0 &= \chi_{(1,1)} + \chi_{(5,5)} + \chi_{(11,11)} + \chi_{(7,7)} + \mu \text{ and } \mu^2 \text{ rotations,} \\ \chi_1 &= \chi_{(1,7)} + \chi_{(7,1)} + \chi_{(5,8)} + \chi_{(8,5)} + \mu \text{ and } \mu^2 \text{ rotations.}\end{aligned}\tag{3.17}$$

Figure 4: The graph \mathcal{E}_{24} corresponding to the level 21 $su(3)$ modular invariant E_{24} .

One then computes, via an S and a T^{-1} modular transformation, that

$$Z_{g,e} = \chi_0^* \chi_1 + \chi_1^* \chi_0, \quad Z_{g,g} = -i \chi_0^* \chi_1 + i \chi_1^* \chi_0. \quad (3.18)$$

One may note that, as follows from (3.14), no cocycle is required to compute these partition functions. A posteriori, one checks that the relations (3.14) are indeed verified provided $\omega_e = 1$, $\omega_g(g, g) = -1$ and all other values equal to 1. This cocycle is universal for projective \mathbb{Z}_2 actions (in the A_{n-1} models of $su(2)$ for n odd, see Table 1, and in the $E_8^{(*)}$ models of $su(3)$).

4 Proofs

We give in this section the elements needed to prove the results announced in Tables 1 to 3, concerning the maximal symmetry of each model and the corresponding partition functions.

Modular transformations form clearly the most important ingredient. Affine characters $\chi_p(\tau)$, labelled by some finite set P_{++} , transform linearly under the modular group. Under the two fundamental transformations, generating the whole group, the characters transform as [10]

$$\chi_p\left(\frac{-1}{\tau}\right) = \sum_{p' \in P_{++}} S_{p,p'} \chi_{p'}(\tau), \quad \chi_p(\tau + 1) = \sum_{p' \in P_{++}} T_{p,p'} \chi_{p'}(\tau). \quad (4.1)$$

The two matrices S and T , symmetric and unitary, are the essential tools to compute the modular transformations of the partition functions. They generate a representation of (in general) the double covering $SL_2(\mathbb{Z})$ of the modular group. Concrete expressions for S and T are given in [10] for all affine Lie algebras, and will be reproduced below in the case of $\widehat{su}(2)$ and $\widehat{su}(3)$. These two matrices, especially S , possess fascinating and useful symmetry properties under the action of the universal Galois group $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ [35]. We will use them on several occasions, mainly in isolated computer-assisted cases.

4.1 The $su(2)$ theories

The integrable representations of the affine Lie algebra $\widehat{su}(2)_k$, with k the level, a non-negative integer, can be labelled by $su(2)$ highest weights p in $P_{++}^{(n)} = \{p \in \mathbb{Z} : 1 \leq p \leq n-1\}$ where the level has been traded for the height $n = k + 2$. The matrices S and T are given explicitly by

$$S_{p,p'} = \sqrt{\frac{2}{n}} \sin \frac{\pi p p'}{n}, \quad T_{p,p'} = e^{2i\pi(\frac{p^2}{4n} - \frac{1}{8})} \delta_{p,p'}. \quad (4.2)$$

By definition, simple currents correspond to those weights J such that $S_{1,J} = S_{1,1}$. In affine $su(2)$ theories, there are two of them, given by $J_0 = 1$ and $J_1 = n - 1$. The simple current J_1 generates the second order automorphism μ of $P_{++}^{(n)}$, given $\mu(p) = n - p$, with respect to which the matrix S transforms as

$$S_{\mu^k(p), \mu^\ell(p')} = (-1)^{k(p'+1) + \ell(p+1) + k\ell n} S_{p,p'}, \quad a, b = 0, 1. \quad (4.3)$$

The proof of the results in Table 1 follows closely the one given in [8] for the same problem in the minimal models (and is even simpler). In particular, the arguments there show that the $su(2)$ theories have a maximal symmetry at most equal to \mathbb{Z}_2 , except the D_{4m+2} models whose maximal symmetry is a subgroup of \mathbb{Z}_{30} , this last result making use of the Galois symmetry of S . The rest of the proof can be easily adapted from [8]. As illustration, we give the detailed proof for the D_{4m+4} theories. The same method can be used for the corresponding series $D_n^{(*)}$, $n \not\equiv 0 \pmod{3}$, of $su(3)$, and for the diagonal theories, or indeed for any simple current automorphism modular invariant.

The starting point is the modular invariant $D_{\frac{n}{2}+1}$, with $n = 0 \pmod{4}$,

$$Z_{e,e} = \sum_p \chi_p^* \chi_{\mu^{p+1}(p)} = \sum_{p=1, \text{ odd}}^{n-1} |\chi_p|^2 + \sum_{p=2, \text{ even}}^{n-2} \chi_p^* \chi_{n-p}, \quad (4.4)$$

which we suppose is compatible with a \mathbb{Z}_N symmetry.

The matrix $M_{p,p'}^{(e)}(e)$ is a permutation matrix and thus $M_{p,p'}^{(e)}(g)$, specifying the action of a \mathbb{Z}_N generator in the periodic sector, is a phased permutation, with entries equal to N -th roots of 1. Whatever these phases are, its $2N$ -th power is equal to the identity. The modular transformation

$$M^{(g)}(e) = S^\dagger M^{(e)}(g) S, \quad (4.5)$$

shows that the same is true of $M^{(g)}(e)$ which, being positive integer-valued, must be a permutation matrix. Let $M_{p,p'}^{(g)}(e) = \delta_{p,\pi(p')}$.

The entry $p = p' = 1$ of the previous equation, written as $S M^{(g)}(e) = M^{(e)}(g) S$, shows that $\pi(1) = n - 1$ must be the non-trivial simple current ($\pi(1)$ cannot be 1, because the twisted sector does not contain the identity field). Taking the same equation again for arbitrary p and $p' = 1$, one finds, from (4.3), that the phases are in fact equal to signs, implying $N = 2$. More precisely, one finds the explicit form $M_{p,p'}^{(e)}(g) = (-1)^{p+1} \delta_{p', \mu^{p+1}(p)}$. This in turn determines $M^{(g)}(e)$ as well as $M_{p,p'}^{(g)}(g)$, as given in Table 1.

This shows that \mathbb{Z}_2 is the only cyclic symmetry compatible with the modular invariant (4.4), and that it has a unique realization in the periodic sector, completing the proof that it is the maximal symmetry. ■

4.2 The $su(3)$ theories

The characters of $\widehat{su}(3)_k$ are indexed by $su(3)$ dominant weights in $P_{++}^{(n)} = \{p = (a, b) \in \mathbb{Z}^2 : 1 \leq a, b, a + b \leq n - 1\}$, with the height defined by $n = k + 3$. The modular matrices read

$$S_{p,p'} = \frac{-i}{\sqrt{3}n} \sum_{w \in W(su(3))} (\det w) e^{-2i\pi(w(p) \cdot p')/n}, \quad (4.6)$$

$$T_{p,p'} = e^{2i\pi(a^2+b^2+ab-n)/3n} \delta_{p,p'}, \quad (4.7)$$

where the w summation is over the Weyl group of $su(3)$. They satisfy $S^2 = (ST)^3 = C$, where the CFT charge conjugation coincides with $su(3)$ conjugation, $C(a, b) = (b, a)$. In particular $S_{p,p'}^* = S_{C(p),p'} = S_{p,C(p')}$.

When one of the indices of S is a diagonal weight, the expression simplifies to

$$S_{(l,l),(a,b)} = \frac{8}{n\sqrt{3}} \sin\left[\frac{\pi al}{n}\right] \sin\left[\frac{\pi bl}{n}\right] \sin\left[\frac{\pi(a+b)l}{n}\right]. \quad (4.8)$$

There are three simple currents J , satisfying as before $S_{(1,1),J} = S_{(1,1),(1,1)}$, given by $J_0 = (1, 1)$, $J_1 = (n - 2, 1)$ and $J_2 = (1, n - 2)$. The last two generate order 3 automorphisms of $P_{++}^{(n)}$, given by $\mu(a, b) = (n - a - b, a)$ and $\mu^2(a, b) = (b, n - a - b)$, under which S transforms as

$$S_{\mu^k(p), \mu^\ell(p')} = e^{2i\pi(kt(p') + \ell t(p) + k\ell n)/3} S_{p,p'}, \quad k, \ell = 0, 1, 2, \quad (4.9)$$

with $t(a, b) = a - b \bmod 3$ the triality.

The automorphism modular invariants of $su(3)$ can be handled by the method detailed in the previous section in the case of $su(2)$, while the few exceptional invariants can be analyzed on a case-by-case basis, with the results given in Table 2. The remaining series $D_n^{(*)}$, with $n = 0 \bmod 3$, is more peculiar and must be treated separately. It could in principle be handled by the same methods as in [8], that relied on the explicit solution of the constraints imposed by the Galois symmetries of S . However, in $su(3)$, this method is tedious, and is in any way useless in other cases. Therefore we have chosen a proof that is independent of Galois arguments.

The rest of this section is devoted to the proof that the $D_n^{(*)}$ modular invariants, $n = 0 \bmod 3$, are compatible with the cyclic symmetry \mathbb{Z}_3 , and only that one if $n \geq 9$, realized in a unique way if $n \geq 12$, and in two different ways if $n = 6, 9$.

The modular invariant partition function reads

$$Z_{0,0} = \sum_{p: t(p)=0} \left[\sum_{j=0}^2 \chi_p^* \chi_{\mu^j(p)} \right] = \frac{1}{3} \sum_{p: t(p)=0} |\chi_p + \chi_{\mu(p)} + \chi_{\mu^2(p)}|^2. \quad (4.10)$$

All fields appear with a multiplicity equal to 1, except $[(\frac{n}{3}, \frac{n}{3}), (\frac{n}{3}, \frac{n}{3})]$ which occurs with multiplicity 3.

So we look for cyclic symmetries and assume the compatibility of the above modular invariant with a \mathbb{Z}_N symmetry. We want to show first that N must be equal to 3 if $n \geq 9$. In order to simplify the notations, we write $M^{(i,j)}$ instead of $M^{(g^i)}(g^j)$ for g a generator of \mathbb{Z}_N . We mainly concentrate on $M^{(1,0)}$ and $M^{(0,1)}$.

A preliminary but very useful observation is that $M_{\mu^k(p), \mu^\ell(p')}^{(1,0)} = M_{p,p'}^{(1,0)}$, as a simple consequence of (4.9) and the fact that $M_{p,p'}^{(0,1)}$ is zero for all p, p' of non-zero triality.

The modular relation $SM^{(1,0)} = M^{(0,1)}S$ implies

$$\sum_p S_{(1,1),p} M_{p,(1,1)}^{(1,0)} = \sum_p M_{(1,1),p}^{(0,1)} S_{p,(1,1)} = (1 + 2 \cos \frac{2\pi q}{N}) S_{(1,1),(1,1)} \leq 3 S_{(1,1),(1,1)}. \quad (4.11)$$

Using again the symmetry (4.9) and the condition $M_{(1,1),(1,1)}^{(1,0)} = 0$, and remembering that $S_{(1,1),p} \geq S_{(1,1),(1,1)}$, we see that the only way the above inequality can be satisfied is that the whole column $M_{p,(1,1)}^{(1,0)}$ is equal to zero, except possibly for the entry corresponding to the fixed point $M_{(\frac{n}{3}, \frac{n}{3}), (1,1)}^{(1,0)} \leq 2$.

A non-zero value $M_{(\frac{n}{3}, \frac{n}{3}), (1,1)}^{(1,0)} = m > 0$ has however to satisfy the above inequality, which explicitly requires

$$m \left(\sin \frac{\pi}{3} \right)^2 \sin \frac{2\pi}{3} \leq 3 \left(\sin \frac{\pi}{n} \right)^2 \sin \frac{2\pi}{n}, \quad (4.12)$$

which holds for $n = 6$ and $m = 1$ only. Feeding this back in (4.11) then yields $q = 0$ and no restriction on N at this stage. Thus the isolated case $n = 6$ demands a separate treatment which we will not detail here, the results being summarized in Table 3.

Thus we may assume $M_{p,(1,1)}^{(1,0)} = 0$ for all p (the arguments that follow are valid for all $n \geq 6$). Then Eq. (4.11) forces $\cos \frac{2\pi q}{N} = -\frac{1}{2}$, which implies N divisible by 3, and hence, for all p of zero triality,

$$\sum_{p'} S_{p,p'} M_{p',(1,1)}^{(1,0)} = 0 = \sum_{p'} M_{p,p'}^{(0,1)} S_{p',(1,1)}. \quad (4.13)$$

Since $S_{p',(1,1)} \neq 0$, one obtains $\sum_k M_{p, \mu^k(p)}^{(0,1)} = 0$, implying in particular $M_{(\frac{n}{3}, \frac{n}{3}), (\frac{n}{3}, \frac{n}{3})}^{(0,1)} = 0$.

The same arguments with the equation $M^{(1,0)}S^\dagger = S^\dagger M^{(0,1)}$ give similar constraints for the columns of $M^{(0,1)}$, namely that $\sum_k M_{\mu^k(p), p}^{(0,1)} = 0$.

To prove that $N = 3$, one can repeat the same calculations for any matrix $M^{(0,x)}$, with x integer between 1 and $N - 1$. Because g acts on the non-degenerate fields by multiplication by a phase ζ , so does g^x , by a phase equal to ζ^x . Therefore $M_{p,p'}^{(0,x)} = [M_{p,p'}^{(0,1)}]^x$ for all pairs $(p, \mu^j(p))$ and $p \neq (\frac{n}{3}, \frac{n}{3})$. It is now straightforward to see that the constraints which will follow from this are

$$\sum_{k=0}^2 [M_{p, \mu^k(p)}^{(0,1)}]^x = \sum_{k=0}^2 [M_{\mu^k(p), p}^{(0,1)}]^x = 0, \quad \text{for all } 1 \leq x \leq N - 1. \quad (4.14)$$

The previous equations for $x = 1$ show that the three numbers $M_{p,p}^{(0,1)}$, $M_{p,\mu(p)}^{(0,1)}$ and $M_{p,\mu^2(p)}^{(0,1)}$ are the three distinct third roots of unity, up to a global N -th root of 1.

Now if $N \geq 6$, one could take $x = 3$, for which (4.14) is clearly violated. Thus we find that *the only cyclic symmetry compatible with the modular invariants D_n , $n \geq 9$ divisible by 3, is a \mathbb{Z}_3 symmetry.*

The next step is to determine the possible realizations of this \mathbb{Z}_3 group, and in fact to show that there is a unique realization when $n \geq 12$, and two realizations for $n = 6, 9$, related to each other by conjugation C .

From the constraints derived in the previous step, one knows that $M_{(\frac{n}{3}, \frac{n}{3}), (\frac{n}{3}, \frac{n}{3})}^{(0,1)} = 0$ and that the 3-by-3 block of $M^{(0,1)}$ containing the identity field has row and column sums equal to zero, and therefore has one of the four forms

$$(a) = \begin{pmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{pmatrix}, \quad (b) = \begin{pmatrix} 1 & \omega^2 & \omega \\ \omega^2 & \omega & 1 \\ \omega & 1 & \omega^2 \end{pmatrix}, \quad (c) = \begin{pmatrix} 1 & \omega^2 & \omega \\ \omega & 1 & \omega^2 \\ \omega^2 & \omega & 1 \end{pmatrix}, \quad (d) = \begin{pmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{pmatrix}, \quad (4.15)$$

where the rows and columns are labelled by $(1, 1)$, $\mu(1, 1)$ and $\mu^2(1, 1)$, and where $\omega = e^{2i\pi/3}$.

We note that the blocks (c) and (d) are related to (a) and (b) respectively by the change $\omega \leftrightarrow \omega^2$, equivalent to a change of generator of \mathbb{Z}_3 . We can therefore omit them. We will show that the form (a) for the block of the identity uniquely determines all \mathbb{Z}_3 partition functions, and that the form (b) leads to a contradiction unless $n = 6, 9$.

Assume that the block of $M^{(0,1)}$ of the identity is (a). Then for any $p \in P_{++}$

$$\sum_{p'} S_{(1,1),p'} M_{p',p}^{(1,0)} = \sum_{p'} M_{(1,1),p'}^{(0,1)} S_{p',p} = [1 + \omega^{1+t(p)} + \omega^{2(1+t(p))}] S_{(1,1),p}. \quad (4.16)$$

The r.h.s. is equal to zero if $t(p) = 0$ or 1, in which case the l.h.s. implies $M_{p',p}^{(1,0)} = 0$ for all weights p such that $t(p) = 0, 1$.

Likewise, the relations

$$\sum_{p'} M_{p,p'}^{(1,0)} S_{p',(1,1)}^\dagger = \sum_{p'} S_{p,p'}^\dagger M_{p',(1,1)}^{(0,1)} = [1 + \omega^{2-t(p)} + \omega^{2(2-t(p))}] S_{p,(1,1)}^\dagger \quad (4.17)$$

show $M_{p,p'}^{(1,0)} = 0$ for all weights p such that $t(p) = 0, 1$.

So altogether, one finds that $M^{(1,0)}$ is non-zero on the triality 2 weights only. For those, the previous two equations yield

$$\sum_{p'} S_{(1,1),p'} M_{p',p}^{(1,0)} = \sum_{p'} M_{p,p'}^{(1,0)} S_{p',(1,1)} = 3S_{(1,1),p}, \quad t(p) = 2. \quad (4.18)$$

They are clearly satisfied if we set $M_{p,p'}^{(1,0)} = 1$ if $p' = p, \mu(p)$ or $\mu^2(p)$, and 0 otherwise. We show that this is in fact the only solution.

It is certainly true for the second fundamental weight $\lambda^2 = (1, 2)$ and its two partners $\mu(\lambda^2)$, $\mu^2(\lambda^2)$, because, in the subset of weights with triality 2, they have the smallest value of $S_{(1,1),p}$ (and so of quantum dimension) [36]. Therefore, $M_{\mu^k(\lambda^2),p}^{(1,0)} = M_{p,\mu^\ell(\lambda^2)}^{(1,0)} = 1$ if $p \in \{\lambda^2, \mu(\lambda^2), \mu^2(\lambda^2)\}$, and 0 if p is anything else.

Writing once more the equation $SM^{(1,0)} = M^{(0,1)}S$ for an arbitrary p of zero triality and λ^2 , one obtains

$$\sum_{p'} S_{p,p'} M_{p',\lambda^2}^{(1,0)} = 3S_{p,\lambda^2} = \sum_{k=0}^2 M_{p,\mu^k(p)}^{(0,1)} S_{\mu^k(p),\lambda^2} = \sum_{k=0}^2 \omega^{2k} M_{p,\mu^k(p)}^{(0,1)} S_{p,\lambda^2}. \quad (4.19)$$

As $S_{p,\lambda^2} \neq 0$ for all $p \neq (\frac{n}{3}, \frac{n}{3})$, one deduces that $M_{p,\mu^k(p)}^{(0,1)} = \omega^k$ for all such p , or in other words, that all 3-by-3 blocks of $M^{(0,1)}$ are the same and equal to the matrix (a). Since we already know that $M_{(\frac{n}{3}, \frac{n}{3})}^{(0,1)} = 0$, one has

$$Z_{e,g} = \frac{1}{3} \sum_{p:t(p)=0} |\chi_p + \omega \chi_{\mu(p)} + \omega^2 \chi_{\mu^2(p)}|^2. \quad (4.20)$$

Its various modular transformations fill the table of partition functions, given in Table 2. Thus, when the block of $M^{(0,1)}$ containing the identity is given by the matrix (a) in (4.15), there is a unique realization of the \mathbb{Z}_3 symmetry, for all $n \geq 6$ (divisible by 3).

There is a second independent possibility for that block, namely (b). Since the first columns of (a) and (b) are equal, Eq. (4.17) remains, while Eq. (4.16) only slightly changes, to the effect that now only the columns of $M^{(1,0)}$ labelled by weights of triality 1 are non-zero, whereas the rows labelled by weights of triality 2 are non-zero. Equivalently, defining $\widetilde{M}_{p,p'}^{(1,0)} = M_{p,C(p')}^{(1,0)}$, one finds that $\widetilde{M}^{(1,0)}$ satisfies all the conditions that $M^{(1,0)}$ satisfied in case (a), namely

$$\sum_{p'} S_{(1,1),p'} \widetilde{M}_{p',p}^{(1,0)} = \sum_{p'} \widetilde{M}_{p,p'}^{(1,0)} S_{p',(1,1)} = 3S_{(1,1),p} \delta_{t(p),2}. \quad (4.21)$$

The same reasoning as above determines the same unique $\widetilde{M}^{(1,0)}$, from which we deduce that the matrix $M^{(1,0)}$ in case (b) is the C -conjugate of that in case (a): $M^{(1,0),(b)} = M^{(1,0),(a)} C$. Then an inverse S modular transformation gives us at once that $M^{(0,1),(b)} = M^{(0,1),(a)} C$, and a partition function $Z_{e,g}$ which is the C -conjugate of that of case (a). This form for $Z_{e,g}$ is not consistent with the modular invariant D_n we started from, unless that invariant is self-conjugate, that is, for $n = 6, 9$.

This concludes our proof for the invariants D_n , $n = 0 \bmod 3$. For $n \geq 12$, there is only one realization of a symmetry \mathbb{Z}_3 , while there are two for $n = 6, 9$, conjugate of each other. Furthermore, there is some room for other cyclic symmetries when $n = 6$, and a separate analysis of this particular case furnishes the results of Section 3.3.1. ■

5 Conclusion

The first purpose of this article was to determine the (finite) symmetries of affine conformal theories based on $su(2)$ and $su(3)$, by using the modular covariance of the torus partition functions. The results, in the form of a list of groups and of partition functions which specify the contents of the twisted sectors and the way these groups act on the fields, have been reported in Section 3.1, see also the Tables 1 to 3.

However a strong motivation for this work was to see if the symmetries present in the field theories, and the representations carried by the various sectors, are in some way encoded in the graphs that have been associated with these theories. Indeed the many points of view that have been taken over the last ten years have consistently shown that these graphs govern many fundamental aspects of those models. Hence our second purpose was to examine and to probe the relevance of the graphs from symmetry considerations.

In this respect, a rather firm conclusion is that, as expected, the graphs indeed have much to say about the symmetries and their realizations in the field theories. This is especially true for the affine $su(2)$ models, where the matching is complete (provided one allows for projective representations). Surprisingly perhaps, this is less so for the $su(3)$ models, where in many cases the graphs have symmetries (automorphisms) unmatched in the field theories. However, taken in the other way, the connection works nicely and universally, since a symmetry in the field theory always has a counterpart in the graph (except for the non-colourable graphs in $su(3)$, for known reasons). In addition the content of the twisted sectors and the way the symmetry is represented in them can be recovered from the graph, thereby extending what the graphs had been devised for in the first place, namely the coding of the diagonal terms in a modular invariant. Most of these features have remained mere observations.

The investigation of the symmetry features of these models in a cylindric geometry would certainly form a natural continuation of this work.

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References

- [1] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. B 241 (1984) 333.
- [2] K. Gawędzki, *Conformal field theory: a case study*, hep-th/9904145.
- [3] M. Walton, *Affine Kac-Moody Algebras and the Wess-Zumino-Witten Model*, hep-th/9911187.

- [4] A. Cappelli, C. Itzykson and J.-B. Zuber, Commun. Math. Phys. 113 (1987) 1.
- [5] A. Kato, Mod. Phys. Lett. A 2 (1987) 585.
- [6] T. Gannon, Annales Inst. Poincaré: Phys. Théor. 65 (1996) 15.
- [7] J.-B. Zuber, Phys. Lett. B 176 (1986) 127.
- [8] P. Ruelle, O. Verhoeven, Nucl. Phys. B 535 (1998) 650.
- [9] P. Ruelle, J. Phys. A: Math. Gen. 32 (1999) 8831.
- [10] V. Kac, *Infinite Dimensional Lie algebras*, (Cambridge University Press, Cambridge, 1990).
- [11] J. Cardy, Nucl. Phys. B 275 [FS17] (1986) 200.
- [12] C. Itzykson and J.-B. Zuber, Nucl. Phys. B 275 [FS17] (1986) 580.
- [13] R.E. Behrend, P.A. Pearce, V.B. Petkova and J.-B. Zuber, *Boundary Conditions in Rational Conformal Field Theories*, **hep-th/9908036**.
- [14] P. Di Francesco, P. Mathieu and D. Sénéchal, *Conformal Field Theory*, (Springer, New York, 1997).
- [15] P. Ginsparg, *Applied conformal field theory*, in *Les Houches, session XLIX, “Fields, Strings and Critical Phenomena”*, eds E. Brézin and J. Zinn-Justin (Elsevier, New York, 1989).
- [16] C. Vafa, Nucl. Phys. B 273 (1986) 592.
- [17] P. Di Francesco and J.-B. Zuber, Nucl. Phys. B 338 (1990) 602.
- [18] P. Di Francesco, Int. J. Mod. Phys. A 7 (1992) 407.
- [19] J. Fuchs and C. Schweigert, Nucl. Phys. B 530 (1998) 99.
- [20] J.-B. Zuber, *CFT, BCFT, ADE and all that*, **hep-th/0006151**.
- [21] G.E. Andrews, R.J. Baxter and J.P. Forrester, J. Stat. Phys. 35 (1984) 193.
- [22] V. Pasquier, Nucl. Phys. B 285 (1987) 162.
- [23] S. Warnaar, B. Nienhuis and K. Seaton, Phys. Rev. Lett. 69 (1992) 710.
- [24] D. O’Brien and P.A. Pearce, J. Phys. A: Math. Gen. 28 (1995) 4891.

- [25] R.E. Behrend and P.A. Pearce, *Integrable and Conformal Boundary Conditions for $\widehat{sl}(2)$ A-D-E Lattice Models and Unitary Minimal Conformal Field Theories*, hep-th/0006094.
- [26] V.B. Petkova and J.-B. Zuber, Nucl. Phys. B 438 (1995) 347; Nucl. Phys. B 463 (1996) 161.
- [27] J. Böckenhauer, D.E. Evans and Y. Kawahigashi, Commun. Math. Phys. 208 (1999), 429; Commun. Math. Phys. 210 (2000) 733.
- [28] J. Cardy, Nucl. Phys. B 324 (1989) 581.
- [29] G. Pradisi, A. Sagnotti and Ya.S. Stanev, Phys. Lett. B 381 (1996) 97.
- [30] J. Fuchs and C. Schweigert, Phys. Lett. B 414 (1997) 251.
- [31] I. Kostov, Nucl. Phys. B 300 (1988) 559.
- [32] Thomas and Wood, *Group Tables*, (Shiva Publishing Limited, 1980).
- [33] G. Karpilovsky, *Projective representations of finite groups*, (Marcel Dekker, New York, 1985).
- [34] F. Xu, Commun. Math. Phys. 192 (1998) 349.
- [35] A. Coste and T. Gannon, Phys. Lett. B 323 (1994) 316.
- [36] T. Gannon, P. Ruelle and M. Walton, Commun. Math. Phys. 179 (1996) 121.